AI-ASSISTED RESEARCH REPORT: FINITE HAUSDORFF HYPERSPACES AND GROMOV–HAUSDORFF GEOMETRY

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Abstract. For a nonempty finite metric space X, let $\mathrm{Exp}(X)$ be its space of nonempty subsets equipped with the Hausdorff metric. We first prove that the hyperspace isometry type determines every finite strongly rigid metric space. Quantitatively, the hyperspace operation preserves the ordinary Gromov–Hausdorff distance in explicit spectral-gap neighborhoods; on each fixed-cardinality stratum the criterion is one-sided. We next prove that Exp is an isometric embedding on the isometry classes of finite ultrametric spaces equipped with the Gromov–Hausdorff ultrametric u_{GH}. The proof combines reconstruction from the hyperspace with commutation under closed ultrametric quotients. These positive results contrast sharply with the ordinary global problem. We give finite ultrametric spaces X_0, Y_0 for which

$$2d_{\mathrm{GH}}(X_0, Y_0) = 3 \quad \text{and} \quad 2d_{\mathrm{GH}}(\mathrm{Exp}(X_0), \mathrm{Exp}(Y_0)) = 2.$$

We then construct equal-cardinality examples, beginning with six points on each side, and an exact scale-quotient criterion explaining the contraction. The resulting families realize every contraction ratio in $[1/2, 1)$. Their distances stabilize after the first hyperspace iteration. A local root-splitting argument shows that this scale-fiber mechanism cannot produce a ratio below $1/2$; whether a universal factor-two lower bound holds for finite ultrametric spaces remains open. An explicit perturbation yields the same strict contraction phenomenon for finite strongly rigid metric spaces with all triangle inequalities strict.

Contents

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1. Introduction

Let $\mathrm{Exp}(X)$ denote the Hausdorff hyperspace of a compact metric space X, that is, the space of its nonempty compact subsets with the Hausdorff metric. Mikhailov proved that the assignment

$$X \longmapsto \mathrm{Exp}(X)$$

is nonexpanding on the Gromov–Hausdorff space [Mik18]. It is therefore natural to ask whether the assignment preserves every Gromov–Hausdorff distance.

There is a substantial exact positive result before one asks for global distance preservation. If X and Y are finite strongly rigid metric spaces, then

$$\mathrm{Exp}(X) \cong \mathrm{Exp}(Y) \quad \Longrightarrow \quad X \cong Y.$$

One proof recognizes singleton subsets from the cardinalities of their closed balls in the hyperspace. A second proof reconstructs the base from any full rainbow family: a family of n hyperspace points whose $\binom{n}{2}$ distances are all different. These mechanisms also give quantitative local results for the ordinary distance. In a symmetric form, Exp preserves $d_{\mathrm{GH}}(X, Y)$ when it is smaller than one quarter of the minimum gap in the two finite distance spectra. For equal-cardinality bases, the gap condition can be imposed on only one of the two spaces. Thus the ordinary hyperspace operation has explicit local distance-preservation regimes even though it is not globally isometric.

The answer depends on which Gromov–Hausdorff geometry is used. On finite ultrametric spaces we consider the Gromov–Hausdorff ultrametric u_{GH}, introduced by Zarichnyi [Zar05]. Our positive result is

$$u_{\mathrm{GH}}(\mathrm{Exp}(U), \mathrm{Exp}(V)) = u_{\mathrm{GH}}(U, V)$$

for all nonempty finite ultrametric spaces U, V. Thus Exp is an isometric embedding in the non-Archimedean setting. The proof has two ingredients. First, a finite ultrametric space can be reconstructed from the isometry type of its Hausdorff hyperspace. Second, taking a Hausdorff hyperspace commutes with every closed quotient of a finite ultrametric space. The structural formula for u_{GH} due to Mémoli, Smith, and Wan [MSW21] then gives the distance equality.

Here "non-Archimedean" refers to the distance u_{GH}, not merely to restricting the ordinary Gromov–Hausdorff distance d_{GH} to ultrametric spaces. This distinction is essential. We construct finite ultrametric spaces X_0, Y_0 such that

$$2d_{\mathrm{GH}}(X_0, Y_0) = 3, \qquad 2d_{\mathrm{GH}}(\mathrm{Exp}(X_0), \mathrm{Exp}(Y_0)) = 2.$$

The mechanism is a failure of Boolean cancellation. The three top components of X_0 all have two points, whereas those of Y_0 have sizes $1, 1, 3$. At the base level a two-point component cannot cover a three-point component below distortion three. After passing to hyperspaces, the seven support blocks have sizes

$$(3, 3, 3, 9, 9, 9, 27) \quad \text{and} \quad (1, 1, 1, 7, 7, 7, 7),$$

and the former blocks cover the latter at distortion two.

The same phenomenon persists at equal cardinality. For every $0 < a < b \leq 2a$ and sufficiently separated top scale H, there are six-point ultrametric spaces X, Y such that

$$2d_{\mathrm{GH}}(X, Y) = b, \qquad 2d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) = a.$$

Appending common singleton top components gives such examples in every cardinality at least six, and varying a/b realizes every contraction ratio in $[1/2, 1)$. The relevant invariant is not the total cardinality of a target support block. At distortion a, only the number of its closed a-classes consumes source capacity. We formulate this observation as a necessary-and-sufficient scale-quotient criterion.

Two boundary facts help delimit this mechanism. First, six points are minimal for hiding a component-level Hall deficit by fixed-scale support capacity. This is a statement about the mechanism, not a classification of all four- and five-point ultrametrics. Second, capacity amplification across an ultrametric split of height r by a source piece of diameter δ is possible only for $\delta < r \leq 2\delta$. Consequently the fixed-scale construction cannot yield a contraction ratio below $1/2$. We do not prove that $1/2$ is a universal lower bound.

Repeated distance levels are not responsible for the global failure. We also give an explicit perturbation consisting of a six-point and a five-point strongly rigid metric space. Every positive distance in each space is distinct and every triangle inequality is strict, while

$$d_{\mathrm{GH}}(X, Y) = 1.504, \qquad d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) \leq 1.0075.$$

Consequently, the Hausdorff hyperspace operation is not an isometric embedding for the ordinary Gromov–Hausdorff distance, even on finite metric spaces in general position. This does not contradict strongly rigid injectivity: strict shortening of a positive distance is different from identifying two isometry classes.

After recalling the two Gromov–Hausdorff distances in section 2, we prove strongly rigid injectivity and the two local theorems in section 3. Finite-ultrametric reconstruction and the u_{GH}-isometry theorem follow in section 4. The ordinary-distance counterexamples are established in section 5.

2. Preliminaries

All metric spaces in the sequel are nonempty. If A is a nonempty subset of a metric space X, write

$$d_X(x, A) = \inf_{a \in A} d_X(x, a).$$

For nonempty compact subsets $A, B \subseteq X$, their Hausdorff distance is

$$d_{\mathrm{H}}^X(A, B) = \max\left\{\sup_{a \in A} d_X(a, B), \sup_{b \in B} d_X(b, A)\right\}.$$

The space of all nonempty compact subsets of X, equipped with this metric, is denoted by $\mathrm{Exp}(X)$. When X is finite, $\mathrm{Exp}(X)$ is simply the set of all nonempty subsets of X.

A *correspondence* between compact metric spaces X and Y is a relation $R \subseteq X \times Y$ whose coordinate projections are surjective. Its distortion is

$$\mathrm{dis}(R) = \sup\left\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in R\right\}.$$

We use the standard correspondence formula

$$d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_R \mathrm{dis}(R), \tag{1}$$

where the infimum ranges over all correspondences between X and Y; see, for example, [BBI01, Chapter 7].

Proposition 2.1 (Mikhailov)**.** *For all compact metric spaces X, Y,*

$$d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) \le d_{\mathrm{GH}}(X, Y).$$

Proof. This is the nonexpansion theorem for the Hausdorff mapping [Mik18]. □

A metric u is an *ultrametric* if

$$u(x, z) \le \max\{u(x, y), u(y, z)\}$$

for all x, y, z. If U is ultrametric, then the Hausdorff metric on $\mathrm{Exp}(U)$ is again ultrametric. Indeed, a point of A can be followed through a nearest point of B to one of C, and the strong triangle inequality gives

$$d_{\mathrm{H}}^U(A, C) \le \max\{d_{\mathrm{H}}^U(A, B), d_{\mathrm{H}}^U(B, C)\}.$$

For compact ultrametric spaces U, V, the Gromov–Hausdorff ultrametric is defined by

$$u_{\mathrm{GH}}(U, V) = \inf_{Z, \varphi_U, \varphi_V} d_{\mathrm{H}}^Z(\varphi_U(U), \varphi_V(V)),$$

where Z is required to be an ultrametric space and φ_U, φ_V are isometric embeddings. This restriction on the ambient metric distinguishes u_{GH} from the ordinary distance d_{GH}.

For $t \geq 0$, let $U_{\mathfrak{c}(t)}$ be the *closed quotient* of an ultrametric space U: points x, x' are identified when $u(x, x') \leq t$. If two equivalence classes are distinct, their distance is the distance between any pair of their representatives; the ultrametric inequality makes this well-defined. For finite ultrametric spaces, Mémoli, Smith, and Wan proved the structural formula

$$u_{\mathrm{GH}}(U, V) = \min\left\{t \geq 0 : U_{\mathfrak{c}(t)} \cong V_{\mathfrak{c}(t)}\right\}. \tag{2}$$

Here and below $\cong$ denotes isometry. We refer to [MSW21, Theorem 3] for equation (2).

3. Strong rigidity and local distance preservation

A finite metric space X is *strongly rigid* if

$$d_X(x, x') = d_X(y, y') > 0 \quad \Longrightarrow \quad \{x, x'\} = \{y, y'\}.$$

Thus all unordered pairs of distinct points determine different distances. Strong rigidity first gives an exact reconstruction result and then, after a quantitative separation argument, local preservation of the ordinary Gromov–Hausdorff distance.

For a nonempty finite metric space M, put

$$\Sigma(M) = \{d_M(x, x') : x, x' \in M\}.$$

Lemma 3.1 (Finite hyperspace distance spectrum)**.** *The distance spectrum of* $\mathrm{Exp}(M)$ *is exactly* $\Sigma(M)$*.*

Proof. Every base distance occurs between a pair of singleton subsets. Conversely, a Hausdorff distance between two finite subsets is the maximum of finitely many nearest-point distances. Each of those nearest-point distances is realized by a pair of points of M, so the maximum belongs to $\Sigma(M)$. □

3.1. Exact reconstruction.

For $A \in \mathrm{Exp}(X)$ and $r \geq 0$, define the closed-ball profile

$$\beta_A^X(r) = \#\{B \in \mathrm{Exp}(X) : d_{\mathrm{H}}^X(A, B) \leq r\}.$$

This profile recognizes the singleton locus intrinsically.

Theorem 3.2 (Singleton recognition)**.** *Let X be a finite strongly rigid metric space with $|X| \geq 3$, and let $A \in \mathrm{Exp}(X)$. Then A is a singleton if and only if $\beta_A^X(r) + 1$ is a power of two for every $r \geq 0$.*

Proof. For a nonempty set $S \subseteq X$, write

$$N_r(S) = \{x \in X : d_X(x, S) \leq r\}.$$

If $A = \{a\}$, then

$$d_{\mathrm{H}}^X(\{a\}, B) = \max_{b \in B} d_X(a, b).$$

Consequently, the sets B counted by $\beta^X_{\{a\}}(r)$ are precisely the nonempty subsets of $N_r(\{a\})$, and hence

$$\beta^X_{\{a\}}(r) + 1 = 2^{|N_r(\{a\})|}.$$

Conversely, suppose $|A| \geq 2$. Among the positive distances having at least one endpoint in A, let R be the largest. Strong rigidity gives a unique pair $\{u, v\}$ realizing R. There is at least one further distance having an endpoint in A, because $|X| \geq 3$; let $r < R$ be the next largest such distance. Every other distance with an endpoint in A is at most r.

Suppose first that exactly one of u, v lies in A, say $u \in A$. Then $N_r(A) = X$. Among the nonempty subsets $B \subseteq X$, exactly $B = \{v\}$ fails the other Hausdorff inclusion $A \subseteq N_r(B)$. Therefore

$$\beta^X_A(r) + 1 = 2^{|X|} - 1,$$

which is not a power of two. If both u and v lie in A, again $N_r(A) = X$, and exactly $B = \{u\}$ and $B = \{v\}$ fail $A \subseteq N_r(B)$. In this case

$$\beta^X_A(r) + 1 = 2^{|X|} - 2 = 2\big(2^{|X|-1} - 1\big),$$

which is not a power of two since $|X| \geq 3$. □

Theorem 3.3 (Strongly rigid hyperspace injectivity)**.** *Let X, Y be nonempty finite strongly rigid metric spaces. If*

$$\mathrm{Exp}(X) \cong \mathrm{Exp}(Y),$$

then $X \cong Y$.

Proof. Hyperspace cardinality gives

$$2^{|X|} - 1 = |\mathrm{Exp}(X)| = |\mathrm{Exp}(Y)| = 2^{|Y|} - 1,$$

so $|X| = |Y|$. The one-point case is immediate. If the common cardinality is two, each hyperspace is an equilateral three-point space whose side length is the unique positive base distance, and the claim follows directly.

Suppose now that the common cardinality is at least three, and let $F : \mathrm{Exp}(X) \to \mathrm{Exp}(Y)$ be an isometry. It preserves every closed-ball profile. By theorem 3.2, it therefore bijects the singleton loci. Write $F(\{x\}) = \{f(x)\}$. Then $f : X \to Y$ is bijective and

$$d_Y(f(x), f(x')) = d^Y_{\mathrm{H}}(F(\{x\}), F(\{x'\})) = d^X_{\mathrm{H}}(\{x\}, \{x'\}) = d_X(x, x').$$

Thus f is an isometry. □

The preceding theorem is an injectivity statement on isometry classes. It does not assert that Exp preserves every ordinary Gromov–Hausdorff distance; section 5 gives strongly rigid pairs for which that stronger statement fails.

3.2. **Spectral-gap neighborhoods.** When M has at least two points, define

$$\eta(M) = \min\{|s-t| : s,t \in \Sigma(M),\ s \neq t\} > 0.$$

By lemma 3.1, this is also the minimum gap between distinct distance levels of $\mathrm{Exp}(M)$.

Theorem 3.4 (Symmetric local distance preservation)**.** *Let X, Y be finite strongly rigid metric spaces with $|X|, |Y| \geq 3$, and put $\eta = \min\{\eta(X), \eta(Y)\}$. If*

$$d_{\mathrm{GH}}(X,Y) < \frac{\eta}{4},$$

then

$$d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) = d_{\mathrm{GH}}(X,Y).$$

Proof. Put

$$g = d_{\mathrm{GH}}(X,Y), \qquad h = d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)).$$

By proposition 2.1, $h \leq g < \eta/4$. Since the hyperspaces are finite, there is an optimal correspondence $R \subseteq \mathrm{Exp}(X) \times \mathrm{Exp}(Y)$ with

$$\delta = \mathrm{dis}(R) = 2h < \frac{\eta}{2}.$$

If one source set were related to two distinct target sets, their positive Hausdorff distance would be at most $\delta < \eta(Y)$, contradicting the definition of $\eta(Y)$. The symmetric argument rules out two distinct source sets related to one target set. Thus R is the graph of a bijection $F : \mathrm{Exp}(X) \to \mathrm{Exp}(Y)$.

Fix a singleton $A \in \mathrm{Exp}(X)$, and let $r \in \Sigma(Y)$ be a nonmaximal distance level with successor r^+. If $d_{\mathrm{H}}^Y(F(A), F(C)) > r$, then

$$d_{\mathrm{H}}^Y(F(A), F(C)) \geq r^+ \geq r + \eta,$$

and the distortion bound gives

$$d_{\mathrm{H}}^X(A, C) \geq r + \eta - \delta > r + \delta.$$

Together with the direct distortion implication, this shows that F identifies

$$\{C : d_{\mathrm{H}}^X(A,C) \leq r + \delta\} \quad \text{with} \quad \{D : d_{\mathrm{H}}^Y(F(A), D) \leq r\}.$$

The same equality of cardinalities is automatic at the maximal level. Closed-ball counts are constant between consecutive spectrum levels, so the profile of $F(A)$, plus one, is a power of two at every radius. Hence theorem 3.2 shows that $F(A)$ is a singleton. Applying the same argument to F^{-1} proves that F bijects the two singleton loci.

The restriction of R to these loci is therefore a base correspondence of distortion at most δ. By equation (1), $g \leq \delta/2 = h$. The reverse inequality is nonexpansion. $\square$

For fixed cardinality the separation can be imposed on only one of the two spaces. The proof uses a second reconstruction mechanism.

Theorem 3.5 (Rainbow-basis reconstruction)**.** *Let Y be a finite strongly rigid metric space with n points. If an n-point subspace*

$$\mathcal{A} = \{A_0, \ldots, A_{n-1}\} \subseteq \mathrm{Exp}(Y)$$

has pairwise distinct distances on its $\binom{n}{2}$ unordered pairs, then $\mathcal{A} \cong Y$.

Proof. The case $n = 1$ is immediate, so assume $n \geq 2$. By lemma 3.1, the $\binom{n}{2}$ distinct positive distances in $\mathcal{A}$ exhaust the positive distance spectrum of Y. Let $D = \mathrm{diam}(Y)$. A pair in $\mathcal{A}$ is at Hausdorff distance D. This distance is realized by a directed nearest-point term, so after interchanging the two sets there are $p \in A$ and $B \in \mathcal{A}$ such that $d_Y(p, B) = D$. Every point of B is then at distance D from p. Strong rigidity gives a unique diameter pair $\{p, q\}$, whence $B = \{q\}$. Relabel so that $A_0 = \{q\}$.

For $i > 0$, put

$$r_i = d_{\mathrm{H}}^{Y}(\{q\}, A_i) = \max_{a \in A_i} d_Y(q, a).$$

These $n - 1$ distinct values exhaust the distances from q. Relabel the remaining sets and write

$$Y = \{q, y_1, \ldots, y_{n-1}\}, \qquad r_1 < \cdots < r_{n-1}, \qquad r_i = d_Y(q, y_i).$$

Strong rigidity makes y_i the unique farthest point from q in A_i, and hence

$$y_i \in A_i, \qquad A_i \subseteq \{q, y_1, \ldots, y_i\}. \tag{3}$$

For $0 \leq k < n$, set

$$\mathcal{A}_k = \{A_0, \ldots, A_k\}, \qquad Y_k = \{q, y_1, \ldots, y_k\}.$$

Every distance inside $\mathcal{A}_k$ is realized by a base edge inside Y_k, by equation (3). Both sets of positive distances have $\binom{k+1}{2}$ members, so

$$\{d_{\mathrm{H}}^{Y}(A_i, A_j) : 0 \leq i < j \leq k\} = \{d_Y(u, v) : u, v \in Y_k,\ u \neq v\}.$$

Subtracting the equality for $k - 1$ shows that, for a permutation π of $\{1, \ldots, k - 1\}$,

$$d_{\mathrm{H}}^{Y}(A_k, A_j) = d_Y(y_k, y_{\pi(j)}) \qquad (1 \leq j < k).$$

The Hausdorff distance on the left is realized by a base edge. Strong rigidity and $y_k \notin A_j$ force $y_{\pi(j)} \in A_j$. Equation equation (3) gives $\pi(j) \leq j$, so π is the identity. Induction on k now yields

$$d_{\mathrm{H}}^{Y}(A_i, A_j) = d_Y(y_i, y_j)$$

for all i, j, with $A_0 \mapsto q$. Thus $\mathcal{A} \cong Y$. $\square$

Theorem 3.6 (One-sided local distance preservation)**.** *Let X, Y be finite strongly rigid metric spaces of the same cardinality, at least two, and put*

$$h = d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)).$$

If $4h < \eta(X)$, then

$$d_{\mathrm{GH}}(X, Y) = h.$$

The same conclusion holds with X and Y interchanged. In particular, the equality holds whenever

$$d_{\mathrm{GH}}(X,Y) < \frac{\eta(X)}{4}$$

or under the analogous condition on Y.

Proof. Let Q be an optimal hyperspace correspondence and put $\delta = \mathrm{dis}(Q) = 2h$. For each $x \in X$, choose $B_x \in \mathrm{Exp}(Y)$ with $(\{x\}, B_x) \in Q$. The sets B_x are distinct: if $B_x = B_{x'}$ for $x \neq x'$, then $d_X(x,x') \leq \delta < \eta(X)$, a contradiction.

If two distinct unordered source pairs determined the same distance between their corresponding B_x's, the distortion inequalities would give

$$|d_X(x,x') - d_X(u,u')| \leq 2\delta = 4h < \eta(X),$$

contradicting strong rigidity and the definition of $\eta(X)$. Hence $\{B_x : x \in X\}$ is a rainbow basis in $\mathrm{Exp}(Y)$. By theorem 3.5, it is isometric to Y. Composing this isometry with $x \mapsto B_x$ gives a bijection $f : X \to Y$ of distortion at most δ. Thus

$$d_{\mathrm{GH}}(X,Y) \leq \frac{\delta}{2} = h.$$

The reverse inequality is proposition 2.1. Finally, if $d_{\mathrm{GH}}(X,Y) < \eta(X)/4$, nonexpansion gives $4h < \eta(X)$. □

Thus, on every fixed-cardinality stratum of finite strongly rigid spaces, Exp preserves the distance from each point throughout an explicit spectral-gap neighborhood. Together with theorem 3.3, this gives local injectivity on that stratum; the quantitative assertion is stronger in that no distance from the chosen center is shortened. This pointwise local statement should not be confused with unrestricted global distance preservation, which is false.

4. The non-Archimedean theorem

We first reconstruct a finite ultrametric space from its Hausdorff hyperspace. Let U be a finite ultrametric space of positive diameter D, and let

$$U = U_1 \sqcup \cdots \sqcup U_k$$

be its open D-partition: two points belong to the same block exactly when their distance is strictly smaller than D. Distinct blocks are at distance D. For every nonempty $S \subseteq \{1, \ldots, k\}$, set

$$E_S(U) = \{A \in \mathrm{Exp}(U) : A \cap U_i \neq \varnothing \iff i \in S\}.$$

These are precisely the open D-blocks of $\mathrm{Exp}(U)$. If $\boxtimes$ denotes the Cartesian product with the maximum metric, then

$$E_S(U) \cong \mathop{\boxtimes}_{i \in S} \mathrm{Exp}(U_i). \tag{4}$$

Indeed, the isometry sends A to $(A \cap U_i)_{i \in S}$. For two sets with the same support, a nearest point can always be chosen in the same top block, and the Hausdorff distance is the maximum of the resulting coordinate distances.

The following elementary algebraic lemma allows us to invert the multiset of all nonempty products in equation (4) without assuming unique factorization.

Lemma 4.1 (Graded subset-product inversion)**.** *Let $\mathscr{M}$ be a commutative monoid with identity* 1*, and suppose there is a map*

$$\deg : \mathscr{M} \longrightarrow \mathbb{N}_{\geq 1}$$

satisfying

$$\deg(ab) = \deg(a)\deg(b), \qquad \deg(a) = 1 \iff a = 1.$$

If finite multisets $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_\ell\}$ satisfy

$$\prod_{i=1}^{k}(1 + a_i) = \prod_{j=1}^{\ell}(1 + b_j) \quad \text{in } \mathbb{N}[\mathscr{M}],$$

then the two multisets are equal.

Proof. Suppose that exactly s of the a_i's equal 1. The coefficient of the identity monomial on the left is 2^s, because no product containing a nonidentity factor can have degree one. The same coefficient on the right therefore recovers the same number of identity factors. Cancel the common scalar 2^s.

Let $d > 1$ be the least degree of a remaining factor. For every $m \in \mathscr{M}$ of degree d, the coefficient of m in the product is exactly the multiplicity with which m occurs as a factor: a product of two or more remaining factors has degree at least $d^2 > d$. Hence the two sides have the same multiset of degree-d factors.

These common factors may be cancelled even when $\mathscr{M}$ is not a cancellative monoid. To see this, let $F \in \mathbb{N}[\mathscr{M}]$ have identity coefficient one and all other monomials of degree greater than one. If $FG = FH$, pass to $\mathbb{Z}[\mathscr{M}]$ and choose the least degree n at which $G - H$ has a nonzero coefficient. For a monomial m of degree n, every contribution to the coefficient of m coming from a nonidentity monomial of F uses a coefficient of $G - H$ of degree strictly smaller than n. Those coefficients vanish, so the coefficient of m in $F(G - H)$ equals its nonzero coefficient in $G - H$, a contradiction. Thus multiplication by F is injective.

Cancel all degree-d factors and repeat. Finiteness terminates the procedure and recovers the complete factor multiset. $\square$

Theorem 4.2 (Finite-ultrametric reconstruction)**.** *Let U, V be nonempty finite ultrametric spaces. If*

$$\operatorname{Exp}(U) \cong \operatorname{Exp}(V),$$

then $U \cong V$.

Proof. We argue by induction on the common cardinality of U and V, which is determined by $|\mathrm{Exp}(U)| = 2^{|U|} - 1$. The one-point case is immediate. Otherwise,

$$\mathrm{diam}(U) = \mathrm{diam}(\mathrm{Exp}(U)) = \mathrm{diam}(\mathrm{Exp}(V)) = \mathrm{diam}(V) = D > 0.$$

Write the open D-partitions as

$$U = U_1 \sqcup \cdots \sqcup U_k, \qquad V = V_1 \sqcup \cdots \sqcup V_\ell.$$

An isometry of the hyperspaces preserves their open D-blocks and the isometry type of each block. By equation (4), equality of these block multisets, after adjoining the empty-support term, is

$$\prod_{i=1}^{k} (1 + [\mathrm{Exp}(U_i)]) = \prod_{j=1}^{\ell} (1 + [\mathrm{Exp}(V_j)])\,. \tag{5}$$

Here brackets denote isometry classes in the commutative monoid of nonempty finite ultrametric spaces under $\boxtimes$. Give this monoid the degree $\deg[W] = |W|$. Its identity is the one-point space, so the hypotheses of lemma 4.1 hold.

The lemma applied to equation (5) matches the factors $\mathrm{Exp}(U_i)$ and $\mathrm{Exp}(V_j)$ up to isometry. Every top block has smaller cardinality than its ambient space. The induction hypothesis therefore gives a matching isometry $U_i \cong V_j$ for every factor. Their union is an isometry $U \cong V$, since distances between distinct top blocks equal D. □

We next compare the closed quotients of U and $\mathrm{Exp}(U)$. Let $q_t : U \to U_{\mathfrak{c}(t)}$ be the quotient map.

Proposition 4.3 (Closed-quotient commutation). *For every nonempty finite ultrametric space U and every $t \geq 0$, the map*

$$Q_t : (\mathrm{Exp}(U))_{\mathfrak{c}(t)} \longrightarrow \mathrm{Exp}(U_{\mathfrak{c}(t)}), \qquad [A] \longmapsto q_t(A),$$

is a natural isometry.

Proof. For nonempty subsets $A, B \subseteq U$,

$$d_{\mathrm{H}}^{U}(A, B) \leq t \quad \Longleftrightarrow \quad q_t(A) = q_t(B). \tag{6}$$

The Hausdorff bound says exactly that each closed t-class met by one set is met by the other. Thus Q_t is well-defined and injective. It is surjective because representatives may be chosen from the members of any nonempty subset of the finite quotient.

If $q_t(A) \neq q_t(B)$, at least one quotient class is met on only one side. Its Hausdorff contribution is greater than t, while the contribution of a class met on both sides is at most t. Distances between distinct classes are their quotient distances. Taking the maximum gives

$$d_{\mathrm{H}}^{U}(A, B) = d_{\mathrm{H}}^{U_{\mathfrak{c}(t)}}(q_t(A), q_t(B)).$$

Together with equation (6), this proves that Q_t is an isometry. □

Theorem 4.4 (Non-Archimedean isometry)**.** *For all nonempty finite ultrametric spaces U, V,*

$$u_{\mathrm{GH}}(\mathrm{Exp}(U), \mathrm{Exp}(V)) = u_{\mathrm{GH}}(U, V).$$

Consequently, Exp *is an isometric embedding on the isometry classes of nonempty finite ultrametric spaces equipped with u_{GH}.*

Proof. The structural formula equation (2), proposition 4.3, and theorem 4.2 give

$$\begin{aligned} u_{\mathrm{GH}}(\mathrm{Exp}(U), \mathrm{Exp}(V)) &= \min\left\{t : (\mathrm{Exp}(U))_{\mathfrak{c}(t)} \cong (\mathrm{Exp}(V))_{\mathfrak{c}(t)}\right\} \\ &= \min\left\{t : \mathrm{Exp}(U_{\mathfrak{c}(t)}) \cong \mathrm{Exp}(V_{\mathfrak{c}(t)})\right\} \\ &= \min\left\{t : U_{\mathfrak{c}(t)} \cong V_{\mathfrak{c}(t)}\right\} \\ &= u_{\mathrm{GH}}(U, V). \end{aligned}$$

The reconstruction theorem also shows directly that the map on isometry classes is injective. □

Remark 4.5. The theorem concerns the Gromov–Hausdorff ultrametric u_{GH}. It makes no claim that the ordinary distance d_{GH} is preserved on ultrametric bases; the next section shows that this stronger-looking interpretation is false.

5. Strict contraction for the ordinary distance

We now show that theorem 4.4 has no analogue for the ordinary Gromov–Hausdorff distance. The first example is ultrametric and has only two positive distance levels in each space.

5.1. **An exact ultrametric counterexample.** Let

$$X_0 = P_1 \sqcup P_2 \sqcup P_3, \qquad |P_1| = |P_2| = |P_3| = 2.$$

Give X_0 the ultrametric whose distance is 2 between distinct points of the same P_i and 10 between points in different components. Let

$$Y_0 = Q_1 \sqcup Q_2 \sqcup Q_3, \qquad (|Q_1|, |Q_2|, |Q_3|) = (1, 1, 3).$$

The three distinct points of Q_3 are at distance 3, and points in different Q_j's are at distance 10.

Theorem 5.1 (Exact ordinary-GH contraction)**.** *The spaces above satisfy*

$$2d_{\mathrm{GH}}(X_0, Y_0) = 3, \qquad 2d_{\mathrm{GH}}(\mathrm{Exp}(X_0), \mathrm{Exp}(Y_0)) = 2.$$

In particular,

$$d_{\mathrm{GH}}(\mathrm{Exp}(X_0), \mathrm{Exp}(Y_0)) < d_{\mathrm{GH}}(X_0, Y_0).$$

Proof. We first compute the base distance. Collapse P_1 and P_2 onto the singleton components Q_1 and Q_2. If $P_3 = \{a, b\}$ and $Q_3 = \{u, v, w\}$, use the component relation

$$\{(a, u), (a, v), (b, w)\}.$$

The first two component relations have distortion 2. The third has distortion 3, attained by the two relation edges with source a, and comparisons between different components compare 10 with 10. Thus there is a base correspondence of distortion 3.

Conversely, suppose that a base correspondence R had distortion strictly less than 3. Any two targets related to source points in one P_i would have distance less than

$$2 + 3 = 5 < 10.$$

They must therefore lie in one target component Q_j. Coverage of Y_0 induces a surjection from the three source components to the three target components, hence a bijection. One two-point component P_i must then cover all three points of Q_3. Some source point is related to two distinct points of Q_3, producing distortion 3, a contradiction. By equation (1), this proves $2d_{\mathrm{GH}}(X_0, Y_0) = 3$.

It remains to compute the hyperspace distance. For every nonempty $S \subseteq \{1, 2, 3\}$, let E_S^X be the family of subsets of X_0 whose set of met components is exactly S. Each $\mathrm{Exp}(P_i)$ is a three-point equilateral space of side 2, and

$$E_S^X \cong \boxtimes_{i \in S} \mathrm{Exp}(P_i).$$

Thus E_S^X is equilateral of side 2, and the seven source-block cardinalities are

$$(3, 3, 3, 9, 9, 9, 27). \tag{7}$$

On the target side, the three support blocks not meeting Q_3 are singletons. Each of the other four blocks is isometric to the seven-point equilateral space $\mathrm{Exp}(Q_3)$ of side 3. The target-block cardinalities are therefore

$$(1, 1, 1, 7, 7, 7, 7). \tag{8}$$

Pair the three size-3 source blocks with the three singleton target blocks. Pair the remaining source blocks, of sizes $9, 9, 9, 27$, with the four size-7 target blocks. In each pair take the graph of a surjection from the source block onto the target block. A size-3 block collapsed onto a singleton has distortion 2. For the other pairs, two distinct source points with the same image cost 2, while two source points with distinct images cost $|2 - 3| = 1$. Every block relation consequently has distortion at most 2.

Different support blocks are at Hausdorff distance 10 on both sides. The union of the seven block relations is therefore a correspondence $\mathcal{Q}$ with $\mathrm{dis}(\mathcal{Q}) \le 2$.

For the reverse bound, the source and target hyperspaces have cardinalities 63 and 31, respectively, and the minimum positive distance in $\mathrm{Exp}(X_0)$ is 2. Given any hyperspace correspondence, choose one target partner for each of the 63 source points. Two distinct source points share a target partner, so those two relation edges have distortion at least 2. Hence every hyperspace correspondence has distortion at least 2, completing the proof. $\square$

Remark 5.2. This example emphasizes the distinction between the two ambient geometries. The closed quotients of X_0 and Y_0 first become isometric at $t = 3$, so $u_{\mathrm{GH}}(X_0, Y_0) = 3$. By theorem 4.4,

$$u_{\mathrm{GH}}(\mathrm{Exp}(X_0), \mathrm{Exp}(Y_0)) = 3$$

even though their ordinary Gromov–Hausdorff distance drops from 3/2 to 1.

5.2. **Scale-quotient capacity.** The previous proof compares the cardinalities of entire support blocks. For multiscale targets, the exact invariant at a prescribed distortion is a quotient cardinality. If Z is a finite ultrametric space and $a \geq 0$, write

$$c_a(Z) = \#\big(Z/(d \leq a)\big).$$

The relation $d \leq a$ is an equivalence relation by the ultrametric inequality.

Lemma 5.3 (Scale-quotient capacity)**.** *Let $E_P(a)$ be the P-point equilateral space of side $a > 0$, and let Z be a nonempty finite ultrametric space with* $\mathrm{diam}(Z) \leq 2a$*. There is a correspondence of distortion at most a between $E_P(a)$ and Z if and only if*

$$P \geq c_a(Z).$$

Proof. In a correspondence of distortion at most a, all target partners of one source point have mutual distance at most a. They therefore lie in one closed a-class. Covering all classes requires $P \geq c_a(Z)$.

Conversely, map the P source points onto the closed a-classes of Z, and relate each source point to every point in its assigned class. Target distances within a class are at most a, while target distances between distinct classes belong to $(a, 2a]$. Comparison with source distances 0 and a gives distortion at most a. ☐

We apply the lemma support by support. Let X have k top components

$$X_i = E_{p_i}(a),$$

at common mutual distance H. Let Y have k nonempty finite-ultrametric top components Y_j, also at mutual distance H, and assume

$$\mathrm{diam}(Y_j) \leq 2a, \qquad H > 2a.$$

For nonempty supports $S, T \subseteq \{1, \ldots, k\}$, put

$$P_S = \prod_{i \in S} (2^{p_i} - 1), \qquad Q_T(a) = \prod_{j \in T} (2^{c_a(Y_j)} - 1).$$

Theorem 5.4 (Exact scale-support criterion)**.** *There is a correspondence of distortion at most a between* $\mathrm{Exp}(X)$ *and* $\mathrm{Exp}(Y)$ *if and only if the nonempty supports admit a bijection ϕ satisfying*

$$P_S \geq Q_{\phi(S)}(a) \qquad \textit{for every nonempty } S.$$

Proof. The source support block indexed by S is the P_S-point equilateral space of side a. The target support block indexed by T is the maximum product of the spaces $\mathrm{Exp}(Y_j)$, $j \in T$. It has diameter at most $2a$.

For a nonempty subset $A \subseteq Z$, its closed a-class in $\mathrm{Exp}(Z)$ is determined by the collection of closed a-classes of Z met by A. Hence

$$c_a(\mathrm{Exp}(Z)) = 2^{c_a(Z)} - 1.$$

Closed quotient cardinalities multiply under maximum products, so the target support block has $Q_T(a)$ closed a-classes.

A source support block cannot meet two target support blocks under a distortion-a correspondence: its internal diameter is at most a, whereas distinct target supports are at distance $H > 2a$. Coverage induces a surjection on the two sets of $2^k - 1$ nonempty supports, hence a bijection. Restriction to a paired block and lemma 5.3 give the required inequalities.

Conversely, use lemma 5.3 on each paired pair of support blocks and take the union of the resulting correspondences. Distances between distinct support blocks are H on both sides. ☐

5.3. **Equal-cardinality counterexamples.** The quotient criterion permits low-scale target fibers without charging their full cardinality. This removes the cardinality imbalance in the first counterexample.

Theorem 5.5 (Equal-cardinality contraction family)**.** *Let $2 \leq s \leq r$, put $n = r + s - 2$, and assume*

$$0 < a < b \leq 2a, \qquad H \geq 2b.$$

Let X have three top components of sizes

$$(s, r, n),$$

all equilateral of side a. Let Y have top-component sizes

$$(s, r - 1, n + 1),$$

where the first two components are equilateral of side a and the last is equilateral of side b. Put distance H between distinct top components on both sides. Then $|X| = |Y|$ and

$$2d_{\mathrm{GH}}(X, Y) = b, \qquad 2d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) = a.$$

Proof. Write $A_j = 2^j - 1$. A correspondence of distortion below b would induce a bijection of the three top components. The target component of size $n+1$ would then have to be covered by a source component of size at most n. Some source point would cover two of its target points, incurring distortion b. The reverse inequality follows by taking full relations between paired top components. Thus the base doubled distance is b.

For the hyperspaces, use the following bijection from source supports to target supports:

$$\{1\} \mapsto \{1\}, \quad \{2\} \mapsto \{2\}, \quad \{3\} \mapsto \{1, 2\}, \quad \{1, 2\} \mapsto \{3\},$$

$$\{1,3\} \mapsto \{1,3\}, \quad \{2,3\} \mapsto \{2,3\}, \quad \{1,2,3\} \mapsto \{1,2,3\}.$$

The target blocks avoiding the third component have closed a-quotient cardinality one. Every target block meeting the third component has closed a-quotient cardinality A_{n+1}. The only nonautomatic source-capacity inequality is

$$A_s A_r \geq A_{n+1} = A_{r+s-1},$$

which follows from

$$A_s A_r - A_{r+s-1} = 2^{r+s-1} - 2^s - 2^r + 2 > 0.$$

The remaining source blocks mapped to supports meeting the third target component contain a factor A_n and at least one factor A_s or A_r; since $s, r \geq 2$, their capacities are at least A_{n+1}. Theorem 5.4 gives hyperspace distortion at most a.

For the reverse inequality, the two hyperspaces have the same cardinality and minimum positive distance a. A correspondence of distortion below a would therefore be a bijection. It would preserve the seven H-support blocks and their cardinalities. Equality of the two support-block cardinality multisets would, by lemma 4.1, force equality of the top-component cardinality multisets. This is impossible because the largest target component has size $n+1$, whereas every source component has size at most n. □

The minimal choice $s = r = 2$ gives two six-point spaces:

$$X = (2_a, 2_a, 2_a), \qquad Y = (2_a, 1, 3_b).$$

Appending the same number of singleton top components to both spaces preserves the proof. Hence every cardinality at least six admits equal-cardinality examples, and varying a/b realizes every ratio in $[1/2, 1)$. Generic sufficiently small perturbations give strongly rigid examples with strict contraction, although the exact displayed distances belong to the symmetric ultrametric models.

5.4. **Iteration and the half-ratio boundary.** The known counterexamples do not continue shrinking under repeated hyperspace formation.

Proposition 5.6 (Stability under iteration)**.** *For the exact equal-cardinality family of theorem 5.5, put*

$$D_m = 2d_{\mathrm{GH}}\big(\mathrm{Exp}^m(X), \mathrm{Exp}^m(Y)\big).$$

Then

$$D_0 = b, \qquad D_m = a \quad (m \geq 1).$$

Proof. The upper bound $D_m \leq a$ follows from the first-iterate construction and nonexpansion. The two iterated spaces have equal cardinality and minimum positive distance a. Thus a correspondence of distortion below a would be a bijection preserving their top support blocks.

For a vector $\mathbf{v} = (v_1, \dots, v_k)$ of top-component cardinalities, the top-component cardinality multiset of its hyperspace is

$$F(\mathbf{v}) = \left\{ \prod_{i \in S} (2^{v_i} - 1) : \varnothing \neq S \subseteq \{1, \dots, k\} \right\}.$$

The graded subset-product inversion lemma says that F is injective. The initial top-size vectors of X and Y differ, so their F^m-images differ for every m. A block-preserving bijection is therefore impossible, proving $D_m \geq a$. ☐

Finally, the threshold $1/2$ has a local explanation. Suppose a target ultrametric space V has diameter $r > \delta$, with open root blocks $V_1, \dots, V_k$. A distortion-δ correspondence between U and V partitions U into nonempty pieces

$$U = U_1 \sqcup \cdots \sqcup U_k$$

such that each U_i corresponds to V_i at distortion δ, and every distance between U_i and U_j, $i \neq j$, belongs to $[r - \delta, r + \delta]$. Indeed, one source point cannot meet two target root blocks, because their mutual distance is $r > \delta$.

Proposition 5.7 (Local half barrier)**.** *A source piece of diameter at most δ can fan out across an ultrametric target split of height $r > \delta$ only when*

$$\delta < r \leq 2\delta.$$

Consequently, the single-scale capacity-amplification mechanism used above cannot produce a contraction ratio below $1/2$.

Proof. If the source piece covers two target root blocks, two of its points compare a source distance at most δ with target distance r. The distortion is at least $r - \delta$, which exceeds δ when $r > 2\delta$. ☐

Question 5.8. Do all nonempty finite ultrametric spaces U, V satisfy

$$2d_{\mathrm{GH}}(\mathrm{Exp}(U), \mathrm{Exp}(V)) \geq \frac{1}{2}\, 2d_{\mathrm{GH}}(U, V)?$$

The proposition answers this only for the fixed-scale fan-out mechanism. A proof for arbitrary finite ultrametrics would have to coordinate support choices through several nested levels. Conversely, a counterexample would have to use genuinely multilevel behavior. We leave both the universal question and the unrestricted four- and five-point equal-cardinality boundary open.

5.5. **A strongly rigid perturbation.** A finite metric space is *strongly rigid* if distinct unordered pairs of distinct points determine distinct distances. We perturb the preceding example so that both spaces are strongly rigid and every triangle inequality is strict.

Label the source points by $0, \dots, 5$. In lexicographic edge order

$$01, 02, 03, 04, 05, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45,$$

define the source distances to be

$$\begin{aligned}(&2.001, 10.002, 10.003, 10.004, 10.005, 10.006, 10.007, 10.008, 10.009,\\ &2.010, 10.011, 10.012, 10.013, 10.014, 2.015).\end{aligned} \tag{9}$$

Label the target points by $0, \dots, 4$. In edge order

$$01, 02, 03, 04, 12, 13, 14, 23, 24, 34,$$

define the target distances to be

$$(10.001, 10.002, 10.003, 10.004, 10.005, 10.006, 10.007, 3.008, 3.009, 3.010). \tag{10}$$

Theorem 5.9 (Strongly rigid contraction)**.** *The metrics X, Y in equations* (9) *and* (10) *are strongly rigid and have only strict triangle inequalities. Moreover,*

$$d_{\mathrm{GH}}(X, Y) = \frac{188}{125} = 1.504, \qquad d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) \leq \frac{403}{400} = 1.0075.$$

Proof. All positive distances in either displayed list are distinct. To check the triangle inequalities for X, regard $\{0, 1\}, \{2, 3\}, \{4, 5\}$ as its three displayed components. A triangle containing two points in one component has one edge at least 2.001 and two cross edges whose difference is at most 0.012. A triangle meeting all three components has all edges between 10.002 and 10.014. Every such triangle is strict. The verification for Y is similar: the component $\{2, 3, 4\}$ has edge lengths $3.008, 3.009, 3.010$, and all cross edges lie between 10.001 and 10.007. Hence both spaces are strongly rigid metrics in general position.

We next compute the base distance. If a correspondence had distortion less than 3.008, all targets related to one of the source pairs $\{0, 1\}, \{2, 3\}, \{4, 5\}$ would lie in a single target component, since

$$2.015 + 3.008 < 10.001.$$

Coverage again induces a bijection between the three displayed source and target components. One source pair must cover the three-point target component $\{2, 3, 4\}$. Repetition of a source point then incurs at least the target separation 3.008. Thus every base correspondence has distortion at least 3.008.

Equality is attained by the relation

$$\{(0, 0), (1, 0), (2, 1), (3, 1), (4, 2), (4, 3), (5, 4)\}.$$

The two edges with source 4 and targets $2, 3$ incur distortion 3.008. The two collapsed pairs cost 2.001 and 2.010, the other within-component errors are smaller than one, and every cross-component error is at most 0.008. Therefore the exact base distortion is 3.008, and equation (1) gives

$$d_{\mathrm{GH}}(X, Y) = \frac{3.008}{2} = 1.504.$$

For the hyperspace upper bound, use the same seven support-block pairing as in the proof of theorem 5.1. The three source blocks of cardinality 3 collapse onto the three singleton target blocks. Their diameters are at most 2.015. Each remaining source block has cardinality at least 9, diameter at most 2.015, and minimum positive distance at least 2.001. Each remaining target block has cardinality 7, diameter at most 3.010, and minimum positive distance at least 3.008. The graph of any surjection between a paired pair of these blocks has distortion at most

$$\max\{2.015, 3.010 - 2.001\} = 2.015.$$

For points in distinct support blocks, the source Hausdorff distance lies in $[10.002, 10.014]$, while the target Hausdorff distance lies in $[10.001, 10.007]$. The cross-block error is therefore at most 0.013. The union of the seven block correspondences has distortion at most 2.015, and hence

$$d_{\mathrm{GH}}(\mathrm{Exp}(X), \mathrm{Exp}(Y)) \leq \frac{2.015}{2} = 1.0075.$$

This is strictly smaller than $d_{\mathrm{GH}}(X, Y)$. □

The examples settle the unrestricted ordinary-distance isometry question negatively. They do not contradict hyperspace injectivity: strict shortening of a positive distance does not identify two isometry classes, and theorem 5.5 shows that unequal base cardinality is not the source of the failure. What survives for d_{GH} is the general nonexpansion in proposition 2.1, equality on special pairs, and the open possibility of a coarse universal reverse inequality.

Use of AI

OpenAI Codex was used in the development of this report for mathematical exploration, proof auditing, finite computation, organization of the research record, and assistance with drafting and preparing the LaTeX manuscript. All mathematical statements included in the report were selected for inclusion by the author. Yoshito Ishiki assumes sole responsibility for the contents of this report, including the correctness of its statements, proofs, computations, references, and exposition.

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