

An interpolation theorem of metrics and geometric properties in spaces of metrics

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Space of Metrics

Definition 1

For a metrizable space X , we denote by $M(X)$ the set of all metrics which generate the same topology as the topology of X .

Definition 2

For a metrizable space X , we define a function $\mathcal{D}_X : M(X)^2 \rightarrow [0, \infty]$ by

$$\mathcal{D}_X(d, e) = \sup_{x, y \in X} |d(x, y) - e(x, y)|.$$

Remark

The function $\mathcal{D}_X$ is a metric on $M(X)$ valued in $[0, \infty]$.

Motivation

In the speaker's research on metric geometry, the speaker has constructed metric spaces not satisfying geometric properties such as the doubling property and the uniformly disconnectedness.

Question

What is a topological distribution of metrics not satisfying geometric properties?

Motivation (analogy)

When the existence of an “interesting” object is proven, we want to know a topological distribution of such objects.

Theorem (K. Weierstrass, 1872)

There exists a continuous nowhere differentiable function on $\mathbb{R}$.

Theorem (S. Banach, S. Mazurkiewicz, 1931)

The set of all continuous nowhere differentiable functions on $[0, 1]$ is typical in $C[0, 1]$, i.e., it contains a dense G_δ subset.

Known theorems on spaces of metrics

There are several results on $M(X)$.

In 1930, Hausdorff proved the following theorem:

Theorem (The Hausdorff extension theorem on metrics)

For every metrizable space X , for every closed subset A of X , and for every $d \in M(A)$, there exists $D \in M(X)$ such that $D|_A = d$.

There is a proof using the Dugundji extension theorem.

The speaker generalizes the Hausdorff extension theorem to an interpolation theorem of metrics preserving $\mathcal{D}_X$.

Results

- An interpolation theorem of metrics preserving $\mathcal{D}_X$ (Theorem 1);
- A dense G_δ theorem of the set of metrics without a transmissible property (Theorem 2);
- A local version of Theorem 2 (Theorem 3).

Main Theorem 1

A family $\{S_i\}_{i \in I}$ of subsets of a topological space X is said to be *discrete* if for every $x \in X$ there exists a neighborhood of x intersecting at most single member of $\{S_i\}_{i \in I}$.

Theorem 1 ([I])

Let X be a metrizable space, and let $\{A_i\}_{i \in I}$ be a discrete family of closed subsets of X . Then for every metric $d \in M(X)$, and for every family $\{e_i\}_{i \in I}$ of metrics with $e_i \in M(A_i)$, there exists a metric $m \in M(X)$ satisfying the following:

- (1) *for every $i \in I$ we have $m|_{A_i^2} = e_i$;*
- (2) *$\mathcal{D}_X(m, d) = \sup_{i \in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2})$.*

Outline of proof of Theorem 1

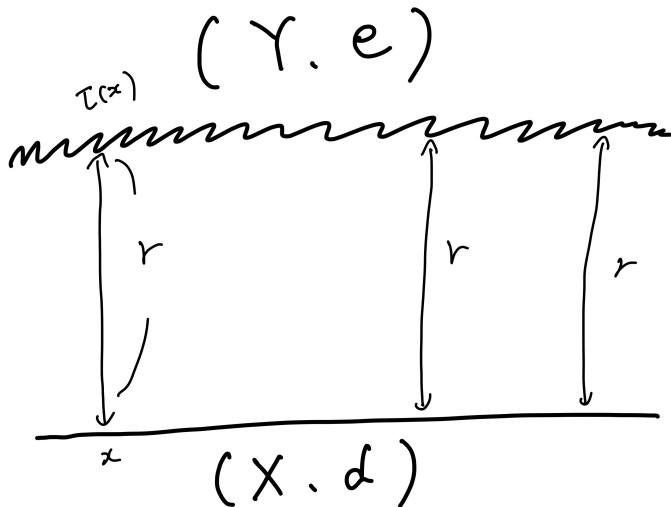
We convert a problem on family of metrics (Thm 1) into a problem on a single metric by using **Amalgamation Lemma**:

Lemma 1

Let X be a metrizable space. Let $d, e \in M(X)$. Let Y be a copy of X , and let $\tau : X \rightarrow Y$ be a trivial bijection. If $\mathcal{D}_X(d, e) \leq r$, then there exists a metric h on $X \sqcup Y$ with $h|_{X^2} = d$, $h|_{Y^2} = e$, and $h(x, \tau(x)) = r$ for all $x \in X$.

Outline of proof of Theorem 1

Picture of Lemma 1.



Outline of proof of Theorem 1

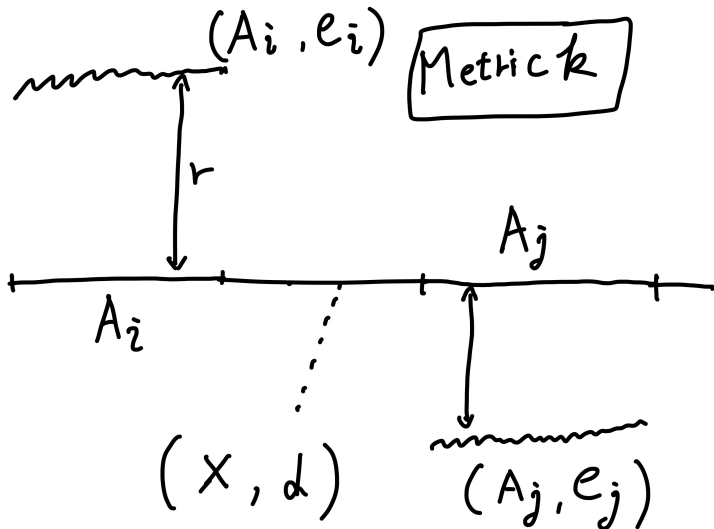
Put $r = \sup_{i \in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2})$.

We join the metric d and the family $\{e_i\}_{i \in \mathbb{N}}$ of metrics.

We obtain a metric k on $X \sqcup \coprod A_i$.

Outline of proof of Theorem 1

Picture of a metric k



Outline of proof of Theorem 1

Let (X, d) be a metric space. For $p \in X$, define a function $d_p : X \rightarrow \mathbb{R}$ by $d_p(x) = d(p, x)$. We denote by $C_b(X)$ the space of all bounded continuous real functions on X with the sup-norm. Then,

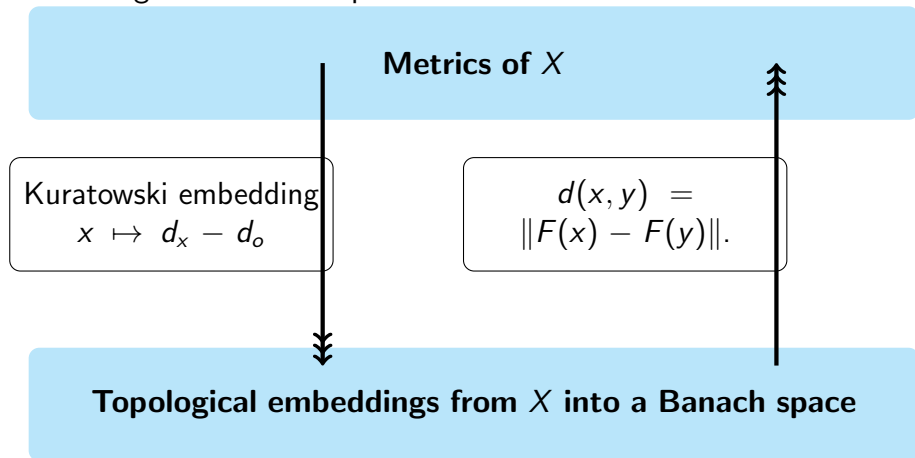
Theorem (The Kuratowski embedding)

For a metric space (X, d) , and for a fixed base point $o \in X$, a map $K : X \rightarrow C_b(X)$ defined by $K(x) = d_x - d_o$ is an isometric embedding.

Let X be a metrizable space, and let B be a Banach space. Let $F : X \rightarrow B$ be a topological embedding. Then, a metric $d : X \times X \rightarrow \mathbb{R}$ defined by $d(x, y) = \|F(x) - F(y)\|$ is in $M(X)$.

Outline of proof of Theorem 1

The following is the correspondence between metrics and topological embeddings into Banach spaces.



Outline of proof of Theorem 1

By the correspondence describe above, we convert Theorem 1 into a problem on interpolation of topological embeddings into a Banach space.

Problem on interpolation of metrics (Thm1)

Kuratowski embedding

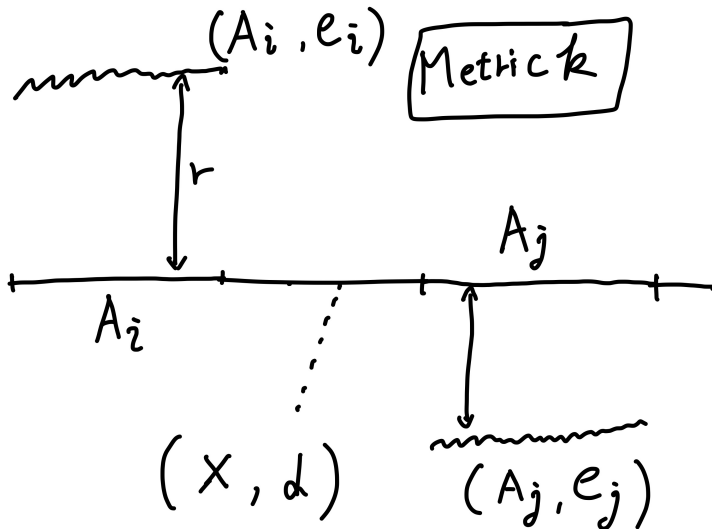
$$x \mapsto d_x - d_o$$

$$d(x, y) = \|F(x) - F(y)\|.$$

Problem on interpolation of topological embeddings into a Banach space

Outline of proof of Theorem 1

Picture of a metric k



Outline of proof of Theorem 1

Let V be a Banach space. We denote by $\mathcal{CC}(V)$ the set of all non-empty closed convex subsets of V .

For a topological space X we say that a map $\phi : X \rightarrow \mathcal{CC}(V)$ is **lower semi-continuous** if for every open subset O of V the set $\{x \in X \mid \phi(x) \cap O \neq \emptyset\}$ is open in X .

The following is known as the Michael continuous selection theorem for paracompact spaces.

Theorem (The Michael continuous selection theorem)

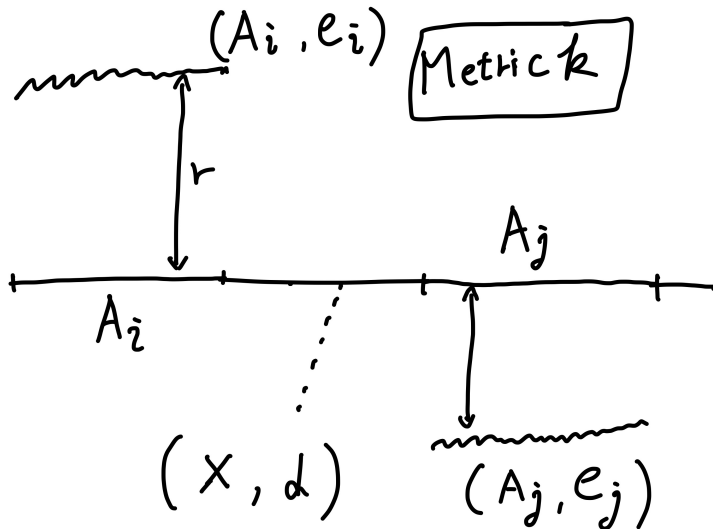
Let X be a paracompact space, and A a closed subsets of X . Let V be a Banach space. Let $\phi : X \rightarrow \mathcal{CC}(V)$ be a lower semi-continuous map. If a continuous map $f : A \rightarrow B$ satisfies $f(x) \in \phi(x)$ for all $x \in A$, then there exists a continuous map $F : X \rightarrow V$ with $F|_A = f$ such that for every $x \in X$ we have $F(x) \in \phi(x)$.

Outline of proof of Theorem 1

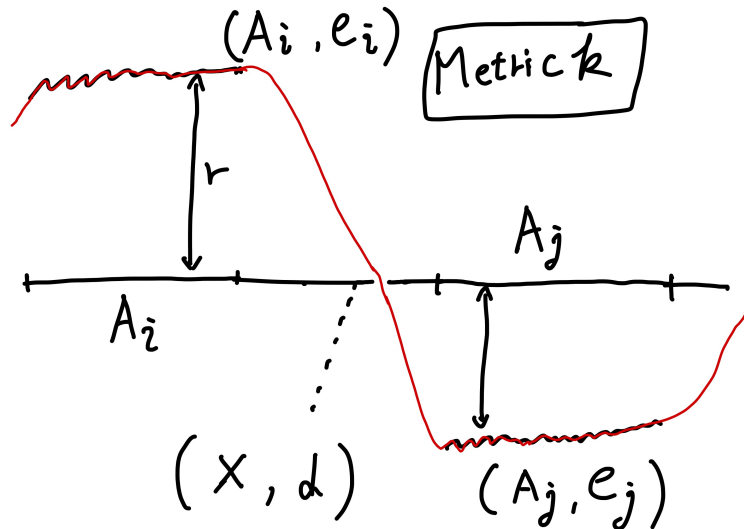
By the Michael continuous selection theorem, we resolve the problem on interpolation of topological embeddings, and we obtain a topological embedding H preserving $\mathcal{D}_X$.

Define $m \in M(X)$ by $m(x, y) = \|H(x) - H(y)\|$, we obtain a desired metric.

Outline of proof of Theorem 1



Outline of proof of Theorem 1



The doubling property

For a metric space (X, d) and for a subset A of X , we set

$$\alpha_d(A) = \inf \{ d(x, y) \mid x, y \in A \text{ and } x \neq y \}.$$

A metric space (X, d) is said to be **doubling** if there exist $C \in (0, \infty)$ and $\alpha \in (0, \infty)$ such that for every finite subset A of X we have

$$\text{card}(A) \leq C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha,$$

where δ_d stands for the diameter.

Remark

A metric space satisfies the doubling property if and only if it has finite Assouad dimension.

The doubling property

Theorem (The Assouad embedding theorem)

A metric space (X, d) is doubling $\iff$ for every $\epsilon \in (0, 1)$, there exist $N \in \mathbb{N}$, $C \geq 1$, $f : X \rightarrow \mathbb{R}^N$ such that

$$C^{-1}d(x, y)^\epsilon \leq d_{\mathbb{R}^N}(f(x), f(y)) \leq Cd(x, y)^\epsilon$$

for all $x, y \in X$

The uniform disconnectedness

A metric space (X, d) is said to be **uniformly disconnected** if there exists $\delta \in (0, 1)$ such that for every finite sequence $\{z_i\}_{i=1}^N$ in X the inequality

$$d(z_i, z_{i+1}) \leq \delta d(z_1, z_N)$$

implies

$$z_1 = z_2 = \cdots = z_N.$$

Remark

A metric space is uniformly disconnected if and only if it is bi-Lipschitz homeomorphic to an ultrametric space.

Metric inequality

There are several geometric inequalities on metric spaces. Let (X, d) be a metric space.

A space (X, d) is an **ultrametric space** $\iff$ it satisfies **the strong triangle inequality**: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

A space (X, d) is a **Ptolemaic metric space** $\iff$ it satisfies **the Ptolemy inequality**: $d(\alpha, \gamma)d(\beta, \delta) \leq d(\alpha, \delta)d(\beta, \gamma) + d(\alpha, \beta)d(\gamma, \delta)$ for all $\alpha, \beta, \gamma, \delta \in X$.

Transmissible properties

The doubling property, the uniform disconnectedness, the strong triangle inequality, and the Ptolemy inequality are geometric properties defined by finite subsets.

The speaker introduced the notion of **transmissible properties**, unifying such a geometric properties.

A transmissible property has a parameter $\mathfrak{G}$. The properties mentioned above are equivalent to $\mathfrak{G}$ -transmissible properties for different **transmissible parameters** $\mathfrak{G}$.

Main results 2

A transmissible parameter $\mathfrak{G}$ is **singular** if there exist a metric d on $\mathbb{N} \cup \{\infty\}$ (one-pt compactification) not satisfy the $\mathfrak{G}$ -transmissible property.

Due to Theorem 1, we obtain the following theorem on dense G_δ subsets in spaces of metrics:

Theorem 2 ([I])

Let $\mathfrak{G}$ be a singular transmissible parameter. For every non-discrete metrizable space X , the set

$\{ d \in M(X) \mid (X, d) \text{ does not satisfies the } \mathfrak{G}\text{-transmissible property} \}$
is dense G_δ in $M(X)$.

Corollary of Theorem 2

The properties (1)–(4) in Example 1 are transmissible properties with singular parameter. Thus Theorem 2 implies:

Corollary 1 ([I])

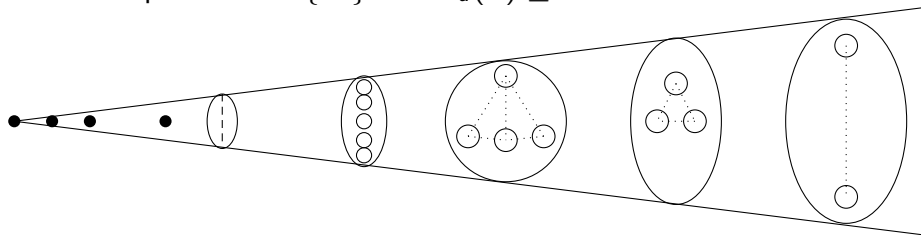
Let X be a non-discrete metrizable space. Then the following sets are dense G_δ in the space $M(X)$ of metrics.

- (1) the set of all metric $d \in M(X)$ for which (X, d) has infinite Assouad dimension;*
- (2) the set of all $d \in M(X)$ for which (X, d) is not bi-Lipschitz embeddable into any ultrametric space;*
- (3) the set of all $d \in M(X)$ for which (X, d) is not an ultrametric;*
- (4) the set of all $d \in M(X)$ for which (X, d) is not a Ptolemaic metric.*

Proof of Theorem 2 for the doubling property

We prove the dense property of the set of non-doubling metrics.
Take arbitrary $d \in M(X)$ and $\epsilon > 0$.

There exists a non-doubling metric e on $\mathbb{N} \cup \{\infty\}$ (one-pt compactification) with diameter $\leq \epsilon$. Take a subset A of X homeomorphic to $\mathbb{N} \cup \{\infty\}$ with $\delta_d(A) \leq \epsilon$.



Proof of Theorem 2 for the doubling property

Theorem 1 in the case where I is a singleton.

Theorem 1'

Let X be a metrizable space, and let A be a closed subset of X . Then for every metric $d \in M(X)$, and for every $e \in M(A)$, there exists a metric $m \in M(X)$ satisfying the following:

- (1) $m|_{A^2} = e$;
- (2) $\mathcal{D}_X(m, d) = \mathcal{D}_A(e, d|_{A^2})$.

Applying Theorem 1', we obtain $m \in M(X)$ such that

- (1) $m|_{A^2} = e$ ($\implies m$ is non-doubling);
- (2) $\mathcal{D}_X(m, d) = \mathcal{D}_A(e, d|_{A^2}) \leq 2\epsilon$.

Proof of Theorem 2 for the doubling property

We conclude that the set

$$\{ d \in M(X) \mid (X, d) \text{ does not satisfy the doubling property} \}$$

is dense in $M(X)$.

We next prove that this set is G_δ .

Proof of Theorem 2 for the doubling property

(X, d) is doubling $\iff$

$$\exists C > 0 \exists \alpha > 0 \forall A : \textit{finite}, \text{card}(A) \leq C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha.$$

Thus,

(X, d) is not doubling $\iff$

$$\forall C \in \mathbb{Q}_+ \forall \alpha \in \mathbb{Q}_+ \exists A : \textit{finite}, \text{card}(A) > C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha.$$

Proof of Theorem 2 for the doubling property

Lemma 2

Let X be a metrizable space. For all $C, \alpha \in \mathbb{Q}_+$, and for every finite subset A of X , the map $\psi : (M(X), \mathcal{D}_X) \rightarrow \mathbb{R}$ defined by

$$\psi(d) = C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha$$

is continuous.

By this lemma, we see that the set

$$S(C, \alpha, A) = \left\{ d \in M(X) \mid \text{card}(A) > C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha \right\}$$

is open for all C, α, A .

Proof of Theorem 2 for the doubling property

Let S be the set of all non-doubling metric on X .

Since (X, d) is not doubling $\iff$

$$\forall C \in \mathbb{Q}_+ \quad \forall \alpha \in \mathbb{Q}_+ \quad \exists A : \textit{finite}, \quad \text{card}(A) > C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha,$$

we have

$$S = \bigcap_{C \in \mathbb{Q}_+} \bigcap_{\alpha \in \mathbb{Q}_+} \bigcup_{A : \textit{finite}} S(C, \alpha, A).$$

Thus S is G_δ .

By a similar method, Theorem 2 for general transmissible properties can be proven.

Main result 3

Theorem 3 ([I])

Let X be a second countable, locally compact locally non-discrete space. Then for every singular transmissible parameter $\mathfrak{G}$, the set

$$\{ d \in M(X) \mid$$

every open metric subspace of (X, d)

$$\text{does not satisfies the } \mathfrak{G}\text{-transmissible property } \}$$

is dense G_δ in $M(X)$.

Corollary of Theorem 3

Theorem 3 implies:

Corollary 2 ([I])

Let X be a second countable locally compact, locally non-discrete metrizable space. Then the following sets are dense G_δ in the space $M(X)$ of metrics.

- (1) the set of all metric $d \in M(X)$ for which all non-empty open metric subspaces of (X, d) have infinite Assouad dimension;*
- (2) the set of all $d \in M(X)$ for which all non-empty open metric subspaces of (X, d) are not bi-Lipschitz embeddable into any ultrametric space;*
- (3) the set of all $d \in M(X)$ for which all non-empty open metric subspaces of (X, d) are not ultrametrics;*
- (4) the set of all $d \in M(X)$ for which all non-empty open metric subspaces of (X, d) are not Ptolemaic metrics.*

Proof of Theorem 3 for the doubling property

Lemma 3

If X is second countable locally compact space, then $M(X)$ is complete metrizable, and hence Baire.

Let $\{U_i\}_{i \in \mathbb{N}}$ a countable open base of X . In the similar way to the proof of Theorem 2, for each $i \in \mathbb{N}$, the set

$$S(U_i) = \{ d \in M(X) \mid (U_i, d|_{U_i^2}) \text{ does not satisfy the doubling property} \}$$

is dense G_δ .

Proof of Theorem 3 for the doubling property

Let

$$S = \{ d \in M(X) \mid \text{every open metric subspace of } (X, d) \\ \text{does not satisfies the doubling property} \}$$

Then, we have

$$S = \bigcap_{i \in \mathbb{N}} S(U_i).$$

Since $M(X)$ is Baire, S is dense G_δ .

Summary

The speaker prove an interpolation theorem of metrics with information of the metric $\mathcal{D}_X$ on $M(X)$ (Theorem 1).

By using Theorem 1, the speaker determine a topological distribution of the set of all metrics without a transmissible property (Theorem 2). A local version of it (Theorem 3) is also proven by argument on Baire spaces.

Thank you very much for your attention.