Typical geometric properties in spaces of metrics

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Baire spaces

Definition 1

A topological space X is said to be a Baire space if the intersection of countably many dense open subsets of X is dense.

Proposition 1

1. *All locally compact Hausdorff spaces are Baire.*
2. *All completely metrizable spaces are Baire.*

Typicality theorem: the set of "singular" objects has a large topological distribution in a suitable space.

Typicality Theorems

Theorem 1 (Banach, Mazurkiwicz)

The set of all continuous nowhere differentiable functions are comeager (contains a dense G_δ subset) in $C[0,1]$.

Theorem 2 (Thom)

Thom's transversality theorem.

Theorem 3 (Chen–Rossi)

Let $\mathcal{K}$ be the space of all non-empty closed subsets of $[0,1]^n$. Let S be the set of all $A \in \mathcal{K}$ such that for every $T \in \mathcal{K}$, and for every $x \in A$ there exists $\lambda \in \mathbb{R}_+$ for which $\lambda \cdot T$ is a tangent cone of A at the point x. Then S is comeager in $\mathcal{K}$.

Space of Metrics

Definition 2

For a metrizable space X, we denote by $\mathrm{M}(X)$ the set of all metrics which generate the same topology as the topology of X.

Definition 3

For a metrizable space X, we define a function $\mathcal{D}_X : \mathrm{M}(X)^2 \to [0, \infty]$ by

$$\mathcal{D}_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|.$$

Remark

The function $\mathcal{D}_X$ is a metric on $\mathrm{M}(X)$ valued in $[0, \infty]$.

Motivation

In the speaker's research on metric geometry, the speaker has constructed metric spaces not satisfying geometric properties such as the doubling property and the uniformly disconnectedness.

Question

What is a topological distribution of metrics not satisfying geometric properties?

Motivation (analogy)

When the existence of an "interesting" object is proven, we want to know a topological distribution of such objects.

Theorem (K. Weierstrass, 1872)

There exists a continuous nowhere differentiable function on $\mathbb{R}$.

Theorem (S. Banach, S. Mazurkiewicz, 1931)

The set of all continuous nowhere differentiable functions on $[0,1]$ is typical in $C[0,1]$, i.e., it contains a dense G_δ subset.

Known theorems on spaces of metrics

There are several result on $\mathrm{M}(X)$.

In 1930, Hausdorff proved the following theorem:

Theorem (The Hausdorff extension theorem on metrics)

For every metrizable space X, for every closed subset A of X, and for every $d \in \mathrm{M}(A)$, there exists $D \in \mathrm{M}(X)$ such that $D|A^2 = d$.

There is a proof using the Dugundji extension theorem.

The speaker generalize the Hausdorff extension theorem to an interpolation theorem of metrics preserving $\mathcal{D}_X$.

Results

- An interpolation theorem of metrics preserving $\mathcal{D}_X$ (Theorem 1);
- A dense G_δ theorem of the set of metrics without a transmissible property (Theorem 2);
- A local version of Theorem 2 (Theorem 3).

Main Theorem 1

A family $\{S_i\}_{i\in I}$ of subsets of a topological space X is said to be *discrete* if for every $x \in X$ there exists a neighborhood of x intersecting at most single member of $\{S_i\}_{i\in I}$.

Theorem 4 ([I])

Let X be a metrizable space, and let $\{A_i\}_{i\in I}$ be a discrete family of closed subsets of X. Then for every metric $d \in \mathrm{M}(X)$, and for every family $\{e_i\}_{i\in I}$ of metrics with $e_i \in \mathrm{M}(A_i)$, there exists a metric $m \in \mathrm{M}(X)$ satisfying the following:

(1) *for every $i \in I$ we have $m|_{A_i^2} = e_i$;*

(2) $\mathcal{D}_X(m, d) = \sup_{i\in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2})$.

Outline of proof of Theorem 1

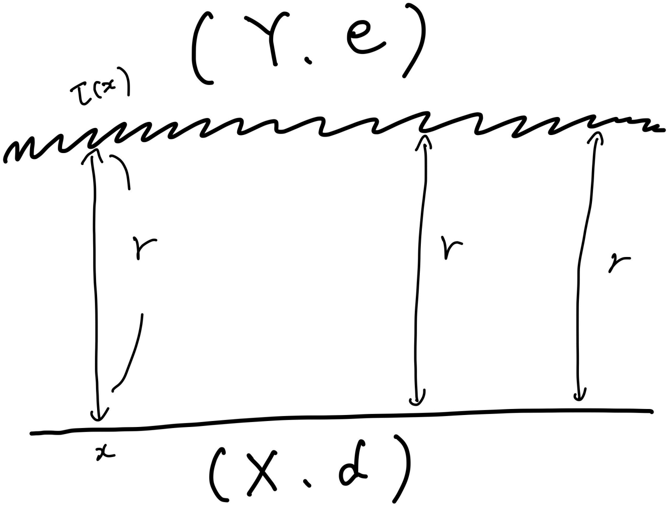
We convert a problem on family of metrics (Thm 1) into a problem on a single metric by using Amalgamation Lemma:

Lemma 1

Let X be a metrizable space. Let $d, e \in \mathrm{M}(X)$. Let Y be a copy of X, and let $\tau : X \to Y$ be a trivial bijection. If $\mathcal{D}_X(d, e) \leq r$, then there exists a metric h on $X \sqcup Y$ with $h|_{X^2} = d$, $h|_{Y^2} = e$, and $h(x, \tau(x)) = r$ for all $x \in X$.

Outline of proof of Theorem 1

Picture of Lemma 1.

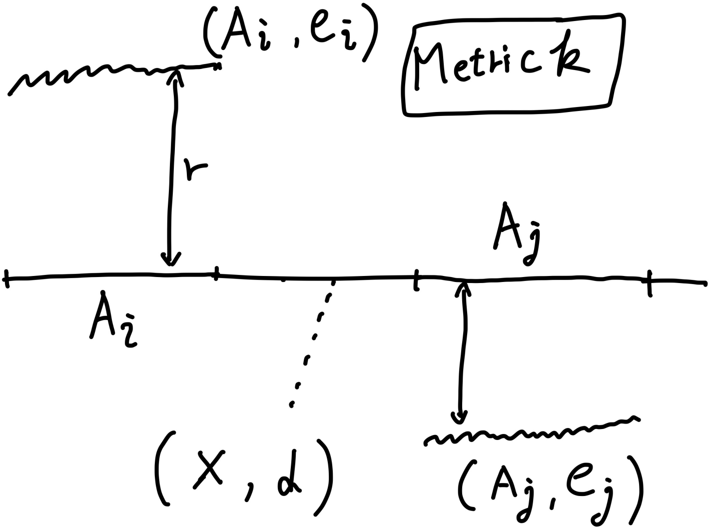


Outline of proof of Theorem 1

Put $r = \sup_{i \in I} \mathcal{D}_{A_i}(e_i, d|_{A_i^2})$.
We join the metric d and the family $\{e_i\}_{i \in \mathbb{N}}$ of metrics.
We obtain a metric k on $X \sqcup \coprod A_i$.

Outline of proof of Theorem 1

Picture of a metric k



Outline of proof of Theorem 1

Let (X, d) be a metric space. For $p \in X$, define a function $d_p : X \to \mathbb{R}$ by $d_p(x) = d(p, x)$. We denote by $C_b(X)$ the space of all bounded continuous real functions on X with the sup-norm. Then,

Theorem (The Kuratowski embedding)

For a metric space (X, d), and for a fixed base point $o \in X$, a map $K : X \to C_b(X)$ defined by $K(x) = d_x - d_o$ is an isometric embedding.

Let X be a metrizable space, and let B be a Banach space. Let $F : X \to B$ be a topological embedding. Then, a metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = \|F(x) - F(y)\|$ is in $\mathrm{M}(X)$.

Outline of proof of Theorem 1

The following is the correspondence between metrics and topological embeddings into Banach spaces.

Metrics of X

Kuratowski embedding
$x \mapsto d_x - d_o$

$d(x, y) = \|F(x) - F(y)\|.$

Topological embeddings from X into a Banach space

Outline of proof of Theorem 1

By the correspondence describe above, we convert Theorem 1 into a problem on interpolation of topological embeddings into a Banach space.

Problem on interpolation of metrics (Thm1)

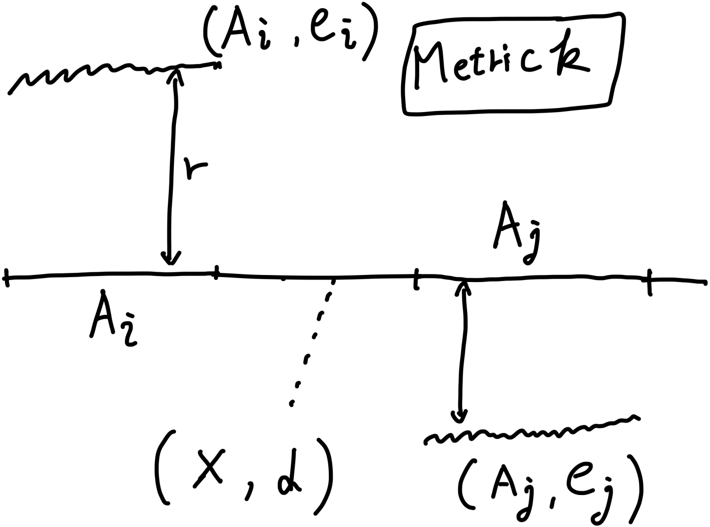
Kuratowski embedding
$x \mapsto d_x - d_o$

$d(x, y) = \|F(x) - F(y)\|.$

Problem on interpolation of topological embeddings into a Banach space

Outline of proof of Theorem 1

Picture of a metric k



Outline of proof of Theorem 1

Let V be a Banach space. We denote by $\mathcal{CC}(V)$ the set of all non-empty closed convex subsets of V.
For a topological space X we say that a map $\phi : X \to \mathcal{CC}(V)$ is lower semi-continuous if for every open subset O of V the set $\{ x \in X \mid \phi(x) \cap O \neq \emptyset \}$ is open in X.
The following is known as the Michael continuous selection theorem for paracompact spaces.

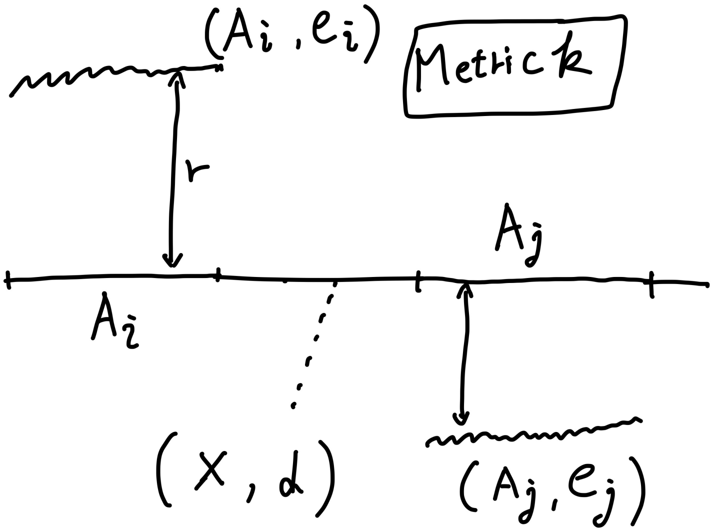
Theorem (The Michael continuous selection theorem)

Let X be a paracompact space, and A a closed subsets of X. Let V be a Banach space. Let $\phi : X \to \mathcal{CC}(V)$ be a lower semi-continuous map. If a continuous map $f : A \to B$ satisfies $f(x) \in \phi(x)$ for all $x \in A$, then there exists a continuous map $F : X \to V$ with $F|_A = f$ such that for every $x \in X$ we have $F(x) \in \phi(x)$.

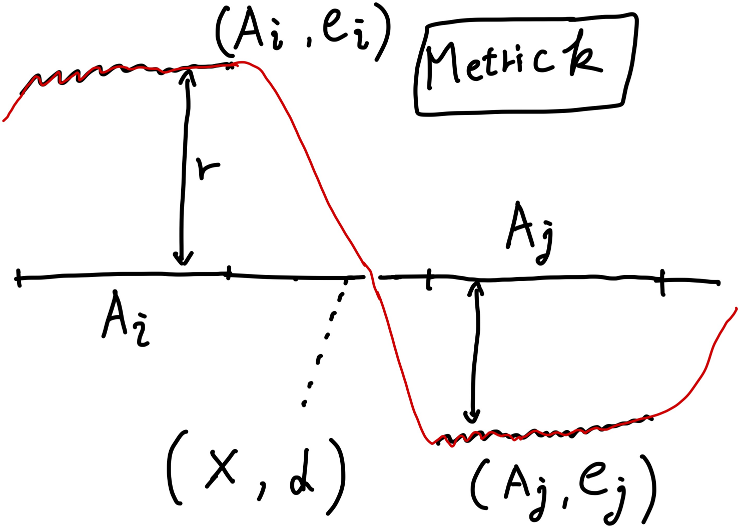
Outline of proof of Theorem 1

By the Michael continuous selection theorem, we resolve the problem on interpolation of topological embeddings, and we obtain a topological embedding H preserving $\mathcal{D}_X$.
Define $m \in \mathrm{M}(X)$ by $m(x, y) = \|H(x) - H(y)\|$, we obtain a desired metric.

Outline of proof of Theorem 1



Outline of proof of Theorem 1



The doubling property

For a metic space (X, d) and for a subset A of X, we set

$$\alpha_d(A) = \inf\{\, d(x, y) \mid x, y \in A \text{ and } x \neq y \,\}.$$

A metric space (X, d) is said to be doubling if there exist $C \in (0, \infty)$ and $\alpha \in (0, \infty)$ such that for every finite subset A of X we have

$$\mathrm{card}(A) \leq C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^{\alpha},$$

where δ_d stands for the diameter.

Remark

A metric space satisfies the doubling property if and only if it has finite Assouad dimension.

The doubling property

Theorem (The Assouad embedding theorem)

A metric space (X, d) *is doubling* $\iff$ *for every* $\epsilon \in (0, 1)$*, there exist* $N \in \mathbb{N}$*,* $C \geq 1$*,* $f : X \to \mathbb{R}^N$ *such that*

$$C^{-1} d(x, y)^{\epsilon} \leq d_{\mathbb{R}^N}(f(x), f(y)) \leq C d(x, y)^{\epsilon}$$

for all $x, y \in X$

The uniform disconnectedness

A metic space (X, d) is said to be uniformly disconnected if there exists $\delta \in (0, 1)$ such that for every finite sequence $\{z_i\}_{i=1}^{N}$ in X the inequality

$$d(z_i, z_{i+1}) \leq \delta d(z_1, z_N)$$

implies

$$z_1 = z_2 = \cdots = z_N.$$

Remark

A metric space is uniformly disconnected if and only if it is bi-Lipschitz homeomorphic to an ultrametric space.

Metric inequality

There are several geometric inequalities on metric spaces. Let (X, d) be a metric space.

A space (X, d) is an ultrametric space $\iff$ it satisfies the strong triangle inequality: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

A space (X, d) is a Ptolemaic metric space $\iff$ it satisfies the Ptolemy inequality: $d(\alpha, \gamma)d(\beta, \delta) \leq d(\alpha, \delta)d(\beta, \gamma) + d(\alpha, \beta)d(\gamma, \delta)$ for all $\alpha, \beta, \gamma, \delta \in X$.

Transmissible properties

The doubling property, the uniform disconnectedness, the strong triangle inequality, and the Ptolemy inequality are geometric properties defined by finite subsets.

The speaker introduced the notion of transmissible properties, unifying such a geometric properties.

A transmissible property has a parameter $\mathfrak{G}$. The properties mentioned above are equivalent to $\mathfrak{G}$-transmissible properties for different transmissible parameters $\mathfrak{G}$.

Main results 2

A transmissible parameter $\mathfrak{G}$ is singular if there exist a metric d on $\mathbb{N} \cup \{\infty\}$(one-pt compactification) not satisfy the $\mathfrak{G}$-transmissible property.

Due to Theorem 1, we obtain the following theorem on dense G_δ subsets in spaces of metrics:

Theorem 5 ([I])

Let $\mathfrak{G}$ be a singular transmissible parameter. For every non-discrete metrizable space X, the set

$$\{\, d \in \mathrm{M}(X) \mid (X, d) \textit{ does not satisfies the } \mathfrak{G}\textit{-transmissible property } \}$$

is dense G_δ in $\mathrm{M}(X)$.

Corollary of Theorem 2

The properties (1)–(4) in Example 1 are transmissible properties with singular parameter. Thus Theorem 2 implies:

Corollary 1 ([I])

Let X be a non-discrete metrizable space. Then the following sets are dense G_δ in the space $\mathrm{M}(X)$ of metrics.

(1) *the set of all metric $d \in \mathrm{M}(X)$ for which (X, d) has infinite Assouad dimension;*

(2) *the set of all $d \in \mathrm{M}(X)$ for which (X, d) is not bi-Lipschitz embeddable into any ultrametric space;*

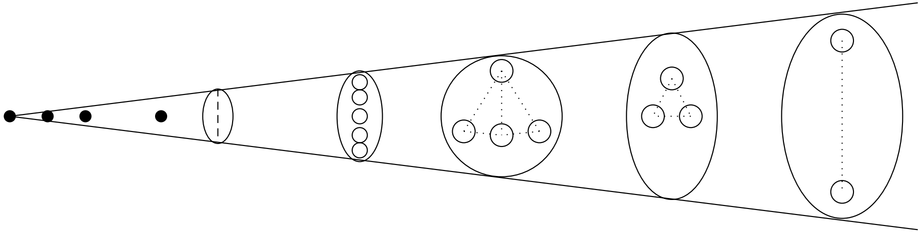
(3) *the set of all $d \in \mathrm{M}(X)$ for which (X, d) is not an ultrametric;*

(4) *the set of all $d \in \mathrm{M}(X)$ for which (X, d) is not a Ptolemaic metric.*

Proof of Theorem 2 for the doubling property

We prove the dense property of the set of non-doubling metrics. Take arbitrary $d \in \mathrm{M}(X)$ and $\epsilon > 0$.

There exists a non-doubling metric e on $\mathbb{N} \cup \{\infty\}$ (one-pt compactification) with diameter $\leq \epsilon$. Take a subset A of X homeomorphic to $\mathbb{N} \cup \{\infty\}$ with $\delta_d(A) \leq \epsilon$.



Proof of Theorem 2 for the doubling property

Theorem 1 in the case where I is a singleton.

Theorem 1'

Let X be a metrizable space, and let A be a closed subset of X. Then for every metric $d \in \mathrm{M}(X)$, and for every $e \in \mathrm{M}(A)$, there exists a metric $m \in \mathrm{M}(X)$ satisfying the following:

(1) $m|_{A^2} = e$;

(2) $\mathcal{D}_X(m, d) = \mathcal{D}_A(e, d|_{A^2})$.

Applying Theorem 1', we obtain $m \in \mathrm{M}(X)$ such that

(1) $m|_{A^2} = e$ ($\Longrightarrow m$ is non-doubling);

(2) $\mathcal{D}_X(m, d) = \mathcal{D}_A(e, d|_{A^2}) \leq 2\epsilon$.

Proof of Theorem 2 for the doubling property

We conclude that the set

$$\{\, d \in \mathrm{M}(X) \mid (X, d) \text{ does not satisfies the doubling property } \}$$

is dense in $\mathrm{M}(X)$.
We next prove that this set is G_δ.

Proof of Theorem 2 for the doubling property

(X, d) is doubling $\iff$

$$\exists C > 0\ \exists \alpha > 0\ \forall\ A : \mathit{finite},\ \mathrm{card}(A) \leq C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^{\alpha}.$$

Thus,
(X, d) is not doubling $\iff$

$$\forall C \in \mathbb{Q}_+\ \forall \alpha \in \mathbb{Q}_+\ \exists\ A : \mathit{finite},\ \mathrm{card}(A) > C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^{\alpha}.$$

Proof of Theorem 2 for the doubling property

Lemma 2

Let X be a metrizable space. For all $C, \alpha \in \mathbb{Q}_+$, and for every finite subset A of X, the map $\psi : (\mathrm{M}(X), \mathcal{D}_X) \to \mathbb{R}$ defined by

$$\psi(d) = C\left(\frac{\delta_d(A)}{\alpha_d(A)}\right)^{\alpha}$$

is continuous.

By this lemma, we see that the set

$$S(C, \alpha, A) = \left\{ d \in \mathrm{M}(X) \mid \mathrm{card}(A) > C\left(\frac{\delta_d(A)}{\alpha_d(A)}\right)^{\alpha} \right\}$$

is open for all C, α, A.

Proof of Theorem 2 for the doubling property

Let S be the set of all non-doubling metric on X.
Since (X, d) is not doubling $\iff$

$$\forall C \in \mathbb{Q}_+ \ \forall \alpha \in \mathbb{Q}_+ \ \exists \ A : \textit{finite}, \ \operatorname{card}(A) > C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^{\alpha},$$

we have

$$S = \bigcap_{C \in \mathbb{Q}_+} \bigcap_{\alpha \in \mathbb{Q}_+} \bigcup_{A : \textit{finite}} S(C, \alpha, A).$$

Thus S is G_δ.
By a similar method, Theorem 2 for general transmissible properties can be proven.

Main result 3

Theorem 6 ([I])

Let X be a second countable, locally compact locally non-discrete space. Then for every singular transmissible parameter $\mathfrak{G}$, the set

$$\{\, d \in \mathrm{M}(X) \mid \text{every open metric subspace of } (X, d) \text{ does not satisfies the } \mathfrak{G}\text{-transmissible property} \,\}$$

is dense G_δ in $\mathrm{M}(X)$.

Corollary of Theorem 3

Theorem 3 implies:

Corollary 2 ([I])

Let X be a second countable locally compact, locally non-discrete metrizable space. Then the following sets are dense G_δ in the space $\mathrm{M}(X)$ *of metrics.*

(1) *the set of all metric $d \in \mathrm{M}(X)$ for which all non-mepty open metric subspaces of (X, d) have infinite Assouad dimension;*

(2) *the set of all $d \in \mathrm{M}(X)$ for which all non-mepty open metric subspaces of (X, d) are not bi-Lipschitz embeddable into any ultrametric space;*

(3) *the set of all $d \in \mathrm{M}(X)$ for which all non-mepty open metric subspaces of (X, d) are not ultrametrics;*

(4) *the set of all $d \in \mathrm{M}(X)$ for which all non-mepty open metric subspaces of (X, d) are not Ptolemaic metrics.*

Proof of Theorem 3 for the doubling property

Lemma 3

If X is second countable locally compact space, then $\mathrm{M}(X)$ is complete metrizable, and hence Baire.

Let $\{U_i\}_{i\in\mathbb{N}}$ a countable open base of X. In the similar way to the proof of Theorem 2, for each $i \in \mathbb{N}$, the set

$$S(U_i) = \{\, d \in \mathrm{M}(X) \mid (U_i, d|_{U_i^2}) \text{ does not satisfy the doubling property} \,\}$$

is dense G_δ.

Proof of Theorem 3 for the doubling property

Let

$$S = \{\, d \in \mathrm{M}(X) \mid \text{every open metric subspace of } (X, d) \text{ does not satisfies the doubling property } \}$$

Then, we have

$$S = \bigcap_{i \in \mathbb{N}} S(U_i).$$

Since $\mathrm{M}(X)$ is Baire, S is dense G_δ.

Summary

The speaker prove an interpolation theorem of metrics with information of the metric $\mathcal{D}_X$ on $\mathrm{M}(X)$ (Theorem 1).

By using Theorem 1, the speaker determine a topological distribution of the set of all metrics without a transmissible property (Theorem 2). A local version of it (Theorem 3) is also proven by argument on Baire spaces.

Thank you very much for your attention.