Quasi-symmetrically invariant properties and spaces of metrics

Yoshito Ishiki

University of Tsukuba

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This Talk consists of two parts.

- Part 1. Cardinality of quasi-symmetric equivalent classes
- Part 2. Dense subsets in spaces of metrics

Part 1
Cardinality of quasi-symmetric equivalent classes

[0] Yoshito Ishiki, Quasi-symmetric invariant properties of Cantor metric spaces, Ann. Inst. Fourier (Grenoble), 69 (6) (2019), 2681–2721.

In the paper [0], the author investigate the cardinality of quasi-symmetric equivalent classes of Cantor metric spaces not satisfying the three condition in the assumption of the David–Semmes theorem.

Metrics

Definition 1

Let X be a set. A function $d : X^2 \to [0, \infty)$ is said to be a metric if the following are satisfied:

(1) *for all $x, y \in X$, we have $d(x, y) = 0$ if and only if $x = y$;*

(2) *for all $x, y \in X$ we have $d(x, y) = d(y, x)$;*

(3) *for all $x, y, z \in Z$ we have $d(x, y) \leq d(x, z) + d(z, y)$.*

Ultrametrics

Definition 2

Let X be a set. A metric d on X is said to be an ultrametric or a non-Archimedean metric if for all $x, y, z \in X$ we have

$$d(x, y) \leq d(x, z) \vee d(z, y),$$

where the symbol $\vee$ stands for the maximum operator on $\mathbb{R}$.

Remark

Let (X, d) be an ultrametric space. Then

- All closed balls are open.
- All open balls are closed.

Therefore every ultrametric space has a clopen base.

Quasi-symmetric maps

Definition 3

An embedding $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is quasi-symmetric (abbreviated as QS) if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that if $x, y, z \in X$ and $t \in [0, \infty)$ satisfy $d_X(x, y) \leq t d_X(x, z)$, then

$$d_Y(f(x), f(y)) \leq \eta(t)\, d_Y(f(x), f(z)).$$

Remark

- The quasi-symmetry gives an equiv. rel. between metric spaces.

Remark

The notion of QS maps was introduced as a generalization of quasi-conformal maps.
The author investigate a geometric aspect of QS maps between metric spaces.

Quasi-symmetric maps

Definition 4

A map $f : (X, d_X) \to (Y, d_Y)$ *between metric spaces is* *homogeneously bi-Hölder* *if there exists* $L \in [1, \infty)$ $\alpha \in (0, \infty)$ *such that for all* $x, y, z \in X$ *we have*

$$L^{-1} d_X(x, y)^\alpha \leq d_Y(f(x), f(y)) \leq L d(x, y)^\alpha.$$

If $\alpha = 1$*, the map is said to be* *bi-Lipschitz*.

Remark

All bi-Lipschitz or homogeneously bi-Höler homeomorphisms are quasi-symmetric.

We next explain three quasi-symmetrically invariant properties.

- The doubling property.
- The uniform disconnectedness.
- The uniform perfectness.

These three properties appear in the assumption of the *David–Semmes Theorem*

1. The doubling property

Definition 5

A metric space (X, d) is doubling (abbreviated as D) if there exists $N \in \mathbb{N}$ such that every closed metric ball with radius r can be covered by at most N closed metric balls with radius $r/2$.

Example 1

- Every finite-dimensional Banach space is doubling.
- Every infinite-dimensional Banach space is not doubling.

Theorem 1 (Assouad Embedding Theorem, 1983)

A metric space (X, d) is doubling if and only if for every $\epsilon \in (0, 1)$ the space (X, d^{ϵ}) can be bi-Lipschitz embedded into $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

The Assouad dimension

Definition 6

For a metric space (X, d), the Assouad dimension $\dim_A(X, d)$ *of (X, d) is the infimum of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every finite subset A of X, we have*

$$\mathrm{Card}(A) \leq C \left(\frac{\max\{d(x, y) \mid x, y \in A\}}{\min\{d(x, y) \mid x \neq y, x, y \in A\}} \right)^{\beta},$$

where Card *stands for the cardinality.*
Note that (X, d) is doubling if $\dim_A(X, d) < \infty$.

Remark

For a metric space (X, d), we have $\dim_H(X, d) \leq \dim_A(X, d)$, where $\dim_H$ is the Hausdorff dimension.

Examples of the Assouad dimension

Example 2

Let $N \in \mathbb{N}$. For every norm $\| * \|$, we have $\dim_A(\mathbb{R}^N, \| * \|) = N$.

Example 3

Let Γ be the Cantor set. Then, $\dim_A(\Gamma, d_{\mathbb{R}}) = \log 2 / \log 3$.

Remark

$\dim_H(X, d) \leq \dim_A(X, d)$.

Theorem 2 ([I])

For every pair $(a, b) \in [0, \infty]^2$ *with* $a \leq b$*,*
there exists a Cantor metric space (X, d) *satisfying*

$$\dim_H(X, d) = a, \quad \dim_A(X, d) = b.$$

2. The uniform disconnectedness

Definition 7

A metric space (X, d) is uniformly disconnected (abbreviated as UD) if there exists $\delta \in (0, 1)$ such that for every non-constant finite sequence $\{z_i\}_{i=1}^{N}$ we have $\delta d(z_1, z_N) < \max_i d(z_i, z_{i+1})$.

Remark

A metric space (X, d) is UD if and only if
(X, d) is bi-Lipschitz homeomorphic to an ultrametric space.

3. The uniform perfectness

Definition 8

A metric space (X, d) *is* *uniformly perfect* *(addreviated as* *UP**) if there exists* $c \in (0, 1]$ *such that for all* $x \in X$ *and* $r \in (0, \mathrm{diam}(X))$ *there exists* $y \in X$ *with*

$$c \cdot r \leq d(x, y) \leq r,$$

where $\mathrm{diam}(X)$ *is the diameter of* (X, d).

Remark

Every connected metric space is uniformly perfect.

Proposition 3

The doubling property, the uniform disconnectedness, and the uniform perfectness are quasi-symmetrically invariant. Namely, if a metric spaces satisfies a property in the three properties, then so do all quasi-symmetric images of the space.

Types of metric spaces

For a property P of metric spaces, if a metric space (X, d) satisfies P, then we put $T_P(X, d) = 1$; otherwise $T_P(X, d) = 0$.

Definition 9

For a triple $(u, v, w) \in \{0, 1\}^3$, a metric space (X, d) has type (u, v, w) if

$$T_D(X, d) = u, \quad T_{UD}(X, d) = v, \quad T_{UP}(X, d) = w.$$

e.g., (X, d) has type $(0, 1, 0) \iff (X, d)$ is non-D, UD, non-UP.

Example 4

The middle-third Cantor set (Γ, d_Γ) has type $(1, 1, 1)$.

The David–Semmes theorem

Theorem 4 (The David-Semmes Theorem, 1997)

If a compact metric space (X, d) is doubling, uniform disconnected, and uniform perfect (type $(1, 1, 1)$), then

$$(X, d) \approx_{QS} (\Gamma . d_\Gamma)$$

where $\approx_{QS}$ means the quasi-symmetric relation.

A Cantor space is standard if it has type $(1, 1, 1)$; otherwise, exotic.

Questions

1. Does there exist an exotic Cantor metric space?
2. Does a uniformization theorem hold for an exotic one?
3. How many/much exotic Cantor metric spaces there exist?

The Brouwer theorem

Theorem 5 (The Brouwer theorem, 1910)

If a compact metric space (X, d) is totally disconnected and perfect, then

$$(X, d) \approx (\Gamma.d_{\Gamma})$$

where $\approx$ means the homeomorphic relation.

Theorem on QS equivalent classes

For a metric space (X, d), the conformal gauge $\mathcal{G}(X, d)$ is defined as

$$\mathcal{G}(X, d) = \{(Y, e) \mid (Y, e) \text{ is a metric space with } (Y, e) \approx_{QS} (X, d)\}.$$

For a triple $(u, v, w) \in \{0, 1\}^3$, we define $\mathcal{M}(u, v, w)$ by

$$\mathcal{M}(u, v, w)$$
$$= \{\mathcal{G}(X, d) \mid (X, d) \text{ is a Cantor metric space of type } (u, v, w)\}.$$

Theorem 6 ([I])

For every $(u, v, w) \in \{0, 1\}$ *except* $(1, 1, 1)$, *we have*

$$\mathrm{Card}(\mathcal{M}(u, v, w)) = 2^{\aleph_0},$$

where Card *denotes the cardinality.*

Remark

The David–Semmes theorem is equivalent to $\mathrm{Card}(\mathcal{M}(1, 1, 1)) = 1$

Theorem 6 answers the questions

Answers

1. Does there exist an exotic Cantor metric space?
   Yes, for each exotic type (u, v, w)

$$\mathrm{Card}(\mathcal{M}(u, v, w)) > 0.$$

2. Does a uniformization theorem hold for an exotic one?
   No, for each exotic type (u, v, w)

$$\mathrm{Card}(\mathcal{M}(u, v, w)) > 1.$$

3. How many/much exotic Cantor metric spaces there exist?
   For each exotic type (u, v, w)

$$\mathrm{Card}(\mathcal{M}(u, v, w)) = 2^{\aleph_0}.$$

Outline of Proof

For a property P of metric spaces, and for a metric space (X, d), we define $S_P(X, d)$ by

$$S_P(X, d) := \{\, x \in X \mid \text{no neighberhood of } x \text{ satisfies } P \,\}.$$

Remark

If $P \in \{D, UD, UP\}$, then

$$(X, d) \approx_{QS} (Y, e) \Longrightarrow S_P(X, d) \approx S_P(Y, e).$$

Definition 10

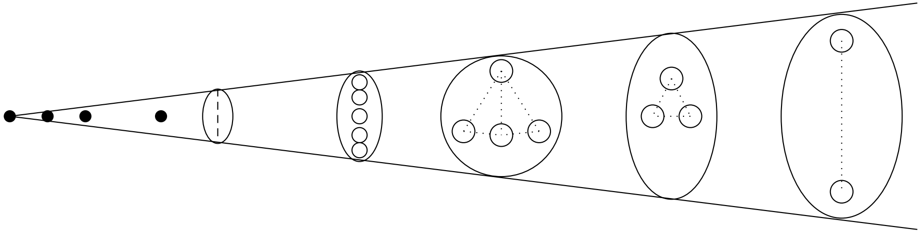
For a property P of metric spaces,
a metric space (X, d) is a P-spike space if $S_P(X, d)$ is a singleton.

Outline of Proof

Proposition ([I])

- *A D-spike Cantor metric space* (E, d_E) *of type (0,1,1) exists.*
- *A UD-spike Cantor metric space* (F, d_F) *of type (1,0,1) exists.*
- *A UP-spike Cantor metric space* (G, d_G) *of type (1,1,0) exists.*

Telescope space constructuion ([I])



Outline of Proof

Proposition ([I])

There exists a continuum family $\{\Xi(x)\}_{x \in I}$ *of closed subsets of the middle-third Cantor set* (Γ, d_Γ) *such that the whole isolated points of* $\Xi(x)$ *is dense in* $\Xi(x)$ *and*

$$x \neq y \Longrightarrow \Xi(x) \not\approx \Xi(y).$$

For each $x \in I$, we obtain Cantor metric spaces

$$(E \times \Xi(x), d_E \times d_x) : (0,1,1) \text{ with } S_D(E \times \Xi(x), d_E \times d_x) = \Xi(x),$$
$$(F \times \Xi(x), d_F \times d_x) : (1,0,1) \text{ with } S_{UD}(F \times \Xi(x), d_F \times d_x) = \Xi(x),$$
$$(G \times \Xi(x), d_G \times d_x) : (1,1,0) \text{ with } S_{UP}(G \times \Xi(x), d_G \times d_x) = \Xi(x).$$

These prove Theorem 6 in the cases of $(0,1,1), (1,0,1), (1,1,0)$.

Outline of Proof

Proposition 7

If a bounded metric space (X, d_X) *has type* (v_1, v_2, v_3)*,*
and if a bounded metric space (Y, d_Y) *has type* (w_1, w_2, w_3)*,*
then the space $(X \sqcup Y, d_X \sqcup d_Y)$ *has type* $(v_1 \wedge w_1, v_2 \wedge w_2, v_3 \wedge w_3)$*.*

Since $(1, 1, 1), (0, 1, 1), (1, 1, 0), (1, 0, 1)$ generate $\{0, 1\}^3$ by $\wedge$, we obtain:

Proposition ([I])

For every $(u, v, w) \in \{0, 1\}^3$ *except* $(1, 1, 1)$*,*
there exists a Cantor metric space of type (u, v, w)*.*

Outline of Proof

By the previous propositions, we can prove Theorem 1 in the remaining cases.

For example, we prove Theorem 1 in the case of $(0,0,0)$.
Let X be a Cantor metric space of type $(1,0,0)$.
For each $x \in I$, we define a Cantor metric space $(H(x), e_x)$ by

$$(H(x), e_x) = ((E \times \Xi(x)) \sqcup X, (d_E \times d_x) \sqcup d_X).$$

Then $S_D(H(x), e_x) = \Xi(x)$. Hence

$$\mathcal{M}(0,0,0) \geq 2^{\aleph_0}.$$

The opposite inequality follows from the separability of Cantor spaces. $\square$

Totally exotic Cantor metric spaces

Concerning the operators S_D, S_{UD} and S_{UP},

Definition 11

For a triple $(u, v, w) \in \{0, 1\}^3$,
a metric space (X, d) has totally exotic type (u, v, w) if

- *(X, d) has type (u, v, w);*
- *$S_P(X, d) = X$ for each failing property $P \in \{D, UD, UP\}$ of (u, v, w).*

Theorem 8 ([I])

For every $(u, v, w) \in \{0, 1\}^3$ except $(1, 1, 1)$,
there exists a Cantor metric space of totally exotic type (u, v, w).

Theorem 8 also states an abundance of exotic Cantor metric space.

Part 2
Dense subsets in spaces of metrics

[1] Yoshito Ishiki, An interpolation of metrics and spaces of metrics, 2020, arXiv 2003.13277.

[2] Yoshito Ishiki, An embedding, an extension, and an interpolation of ultrametrics, arXiv:2008.10209, p-Adic Numbr. Ultr. Anal. Appl., 13 (2021), 117–147.

[3] Yoshito Ishiki, On dense subsets in spaces of metrics, 2021, arXiv:2104.12450.

- In the paper [1], the author investigated topological distributions of geometric properties in spaces of metrics. The author proved that the set of non-doubling metrics and the set of non-doubling metrics are dense G_δ in spaces of metrics.
- In the paper [2], the author proved ultrametric analogues of statements of metrics such as the Arens–Eells embedding theorem, the Hausdorff extension theorem of metrics, and an interpolation theorem of metrics and theorems on topological distribution in the paper [1].
- In the paper [3], the author investigated topological distributions of the doubling property, the uniform disconnectedness, and the uniform perfectness.

Space of Metrics

Definition 12

For a metrizable space X, we denote by $\mathrm{M}(X)$ the set of all metrics which generate the same topology as the topology of X.

Definition 13

For a metrizable space X, we define a function $\mathcal{D}_X : \mathrm{M}(X)^2 \to [0, \infty]$ by

$$\mathcal{D}_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|.$$

Remark

The function $\mathcal{D}_X$ is a metric on $\mathrm{M}(X)$ valued in $[0, \infty]$.

Ultrametrics

Definition 14

Let X be a set. A metric d on X is said to be an ultrametric or a non-Archimedean metric if for all $x, y, z \in X$ we have

$$d(x, y) \leq d(x, z) \vee d(z, y),$$

where the symbol $\vee$ stands for the maximum operator on $\mathbb{R}$. We say that a set S is a range set if S is a subset of $[0, \infty)$ and $0 \in S$. For a range set S, we say that a metric $d : X^2 \to [0, \infty)$ on X is S-valued if $d(X^2)$ is contained in S.

Space of ultrametrics

Definition 15 ([I])

Let S be a range set For a topological space X, we denote by $\mathrm{UM}(X, S)$ *the set of all S-valued ultrametrics which generate the same topology on X.*

Definition 16 ([I])

For a topological space X, and for a range set S, we define a function $\mathcal{UD}_X^S : \mathrm{UM}(X, S)^2 \to [0, \infty]$ *by assigning* $\mathcal{UD}_X^S(d, e)$ *to the infimum of* $\epsilon \in S \sqcup \{\infty\}$ *such that for all $x, y \in X$ we have* $d(x, y) \leq e(x, y) \vee \epsilon$*, and* $e(x, y) \leq d(x, y) \vee \epsilon$*.*

This is an ultrametric analogue of the sup metric. (Replace "$\vee$" with "$+$".)

Let X be a topological space.

- A subset S is said to be F_σ (resp. G_δ) if S is the union of countably many closed subsets of X (resp. the intersection of countably many open subsets of X).
- A subset M is said to be $F_{\sigma\delta}$ (resp. $G_{\delta\sigma}$) if M is the intersection of countably many F_σ subsets of X (resp. the union of countably many G_δ subsets of X).

Remark

Let X be a metrizable space. Then

1. All closed subsets of X are F_σ.
2. All closed subsets of X are G_δ.
3. All open subsets of X are G_δ.
4. All open subsets of X are F_σ.

Baire spaces

Definition 17

A topological space X is said to be a Baire space if the intersection of countably many dense open subsets of X is dense.

Proposition 9

1. *All locally compact Hausdorff spaces are Baire.*
2. *All completely metrizable spaces are Baire.*

Typicality theorem: the set of "singular" objects has a large topological distribution in a suitable space.

Typicality Theorems

Theorem 10 (Banach, Mazurkiwicz)

The set of all continuous nowhere differentiable functions are comeager (contains a dense G_δ subset) in $C[0,1]$.

Theorem 11 (Thom)

Thom's transversality theorem.

Theorem 12 (Chen–Rossi)

Let $\mathcal{K}$ be the space of all non-empty closed subsets of $[0,1]^n$. Let S be the set of all $A \in \mathcal{K}$ such that for every $T \in \mathcal{K}$, and for every $x \in A$ there exists $\lambda \in \mathbb{R}_+$ for which $\lambda \cdot T$ is a tangent cone of A at the point x. Then S is comeager in $\mathcal{K}$.

Typicality theorems on spaces of metrics

The $\mathfrak{G}$-transmissible property is a geometric property on metric spaces defined by finite subsets of metric spaces with transmissible parameter $\mathfrak{G}$.

Theorem 13 ([I])

Let $\mathfrak{G}$ be a singular transmissible parameter. For every non-discrete metrizable space X, the set

$$\{\, d \in \mathrm{M}(X) \mid (X, d) \textit{ does not satisfies the } \mathfrak{G}\textit{-transmissible property} \,\}$$

is dense G_δ in $\mathrm{M}(X)$.

Remark

Note that the doubling property and the uniform disconnectedness are transmissible properties with singular parameters.
The uniform perfectness is not a transmissible property.

Typicality theorems on spaces of metrics

Recall that the doubling property is equivalent to the finiteness of Assouad dimension.

Theorem 14 ([I])

Let X be a non-discrete metrizable space. Then the set

$$\{ d \in \mathrm{M}(X) \mid \dim_A(X, d) = \infty \}$$

is dense G_δ in $\mathrm{M}(X)$.

Recall that the uniform disconnectedness is equivalent to bi-Lipschitz embeddability into Euclidean spaces.

Theorem 15 ([I])

Let X be a non-discrete metrizable space. Then the set

$$\{\, d \in \mathrm{M}(X) \mid (X, d) \textit{ is not bi-Lipschitz embeddable into any ultrametric space } \}$$

is dense G_δ in $\mathrm{M}(X)$.

Typicality theorems on spaces of metrics

We prove Theorem 14 and 15 by using extension theorems of metrics.

Theorem 16 (Hausdorff)

Let X be a metrizable space, A be a closed subset of X. Then for every $e \in \mathrm{M}(A)$, there exists $m \in \mathrm{M}(X)$ such that $m|_{A^2} = e$.

Theorem 17 ([I])

Let X be a metrizable space, and A be a closed subset of X. Let $d \in \mathrm{M}(X)$ and $e \in \mathrm{M}(A)$. Then there exists $m \in \mathrm{M}(X)$ such that

1. $m|_{A^2} = e$;
2. $\mathcal{D}_X(m, d) = \mathcal{D}_A(e, d|_{A^2})$

Typicality theorems on spaces of metrics

Lemma 18

If X is second countable, locally compact metrizable, then $\mathrm{M}(X)$ is Baire.

By using Lemma 18, we can prove a local version of Theorem 14.

Theorem 19 ([I])

Let X be a second countable, locally compact locally non-discrete metrizable space. Then the set

$$\{\, d \in \mathrm{M}(X) \mid \text{for every non-empty open subset } U \text{ we have } \dim_A(U, d) = \infty \,\}$$

is dense G_δ in $\mathrm{M}(X)$.

Typicality theorems on spaces of metrics

The following is a local version of Theorem 15.

Theorem 20 ([I])

Let X be a second countable, locally compact locally non-discrete metrizable space. Then the set

$$\{\, d \in \mathrm{M}(X) \mid \text{for every non-empty open subset } U \text{ the space } (U, d) \text{ is not bi-Lipschitz embeddable into any ultrametric space.} \,\}$$

is dense G_δ in $\mathrm{M}(X)$.

Ultrametrics

The following is an ultrametric analogue of the Hausdorff extension theorem of metrics:

Theorem 21 ([I])

Let S be a range set. Let X be an S-valued ultrametrizable space, A be a closed subset of X. Then for every $e \in \mathrm{M}(A, S)$, there exists $m \in \mathrm{M}(X, S)$ such that $m|_{A^2} = e$.

Ultrametrics

A range set S is said to be quasi-complete if there exists $C \in (0,1]$ such that for every bounded subset E of S there exists $s \in S$ with $\sup E \leq s \leq C \cdot \sup E$.
The following is an ultrametrics version of Theorem 17.

Theorem 22 ([I])

Let S be a C-quasi-complete range set. Let X be an S-valued ultrametrizable space, and A be a closed subset of X. Let $d \in \mathrm{UM}(X,S)$ and $e \in \mathrm{UM}(A,S)$. Then there exists $m \in \mathrm{UM}(X,S)$ such that

1. $m|_{A^2} = e$;
2. $\mathcal{UD}_A^S(e, d|_{A^2}) \leq \mathcal{UD}_X^S(m,d) \leq C \cdot \mathcal{UD}_A^S(e, d|_{A^2})$

Ultrametrics

A range set S has countable coinitiality if there exists $\{r_i\}_{i\in\mathbb{N}}$ in S such that $r_i \to 0$ as $i \to \infty$.
The following is an ultrametric analogue of Theorem 14

Theorem 23 ([I])

Let S be a quasi-complete range set with countable coinitiality. Let X be a non-discrete S-valued ultrametrizable space. Then the set

$$\{\, d \in \mathrm{M}(X) \mid \dim_A(X, d) = \infty \,\}$$

is dense G_δ in $\mathrm{M}(X)$.

Ultrametrics

The following is a local version of Theorem 23.

Theorem 24 ([I])

Let S be a quasi-complete range set with countable coinitiality. Let X be a second countable, locally compact locally non-discrete S-valued ultrametrizable space. Then the set

$$\{\, d \in \mathrm{UM}(X, S) \mid \text{for every non-empty open subset } U \text{ we have } \dim_A(U, d) = \infty \,\}$$

is dense G_δ in $\mathrm{UM}(X, S)$.

Before explaining results in the paper [3], we introduced theorem characterizing topological property of a metrizable space X by the space of metrics $\mathrm{M}(X)$.

Characterizations by spaces of metrics

Theorem 25 (Niemytzki–Tychonoff)

A metrizable space X is compact if and only if all $d \in \mathrm{M}(X)$ are complete.

In the paper [2] the author proved an ultrametric analogue of Niemytzki–Tychonoff theorem.
A range set S has countable coinitiality if there exists $\{r_i\}_{i\in\mathbb{N}}$ in S such that $r_i \to 0$ as $i \to \infty$.

Theorem 26 ([I])

Let S be a range set with the countable coinitiality. Then, an ultrametrizable space X is compact if and only if all $d \in \mathrm{UM}(X, S)$ are complete.

Characterizations by spaces of metrics

Other examples:

Theorem 27 (Dovgoshey–Shcherbak)

An ultrametrizable space X is compact if and only if all $d \in \mathrm{UM}(X)$ are totally bounded.

Theorem 28 (Dovgoshey–Shcherbak)

An ultrametrizable space X is separable if and only if for every $d \in \mathrm{UM}(X)$ we have $\mathrm{Card}(\{ d(x, y) \mid x, y \in X \}) \leq \aleph_0$.

In the paper [3], the author characterize the finite-dimensionality or zero-dimensionality and compactness on metrizable space by the denseness of the set of all doubling or uniformly disconnected metrics in spaces of metrics.

The Covering dimension

We omit the definition of the covering dimension.
In order to explain main results, it is suffices to know the following characterizations.

Proposition 29

A separable metrizable space X is finite-dimensional if and only if X is can be topologically embedded into a Euclidean space.

Proposition 30

A separable metrizable space X is 0-dimensional if and only if $X \neq \emptyset$ and X has a clopen base.

Theorems on denseness

The following is known as the McShane–Whitney extension theorem.

Theorem 31

Let $l \in (0, \infty)$. Let (X, d) be a metric space, and let A be a subset of X. Then, for every l-Lipschitz map $f : (A, d) \to (\mathbb{R}, \| * \|_{\infty})$, there exists an l-Lipschitz map $F : (X, d) \to (\mathbb{R}, \| * \|_{\infty})$ such that $F|_A = f$. Moreover, every l-Lipschitz map $f : (A, d|_{A^2}) \to (\mathbb{R}^n, \| * \|_{\infty})$ can be extended to an l-Lipschitz map from $F : (X, d) \to (\mathbb{R}^n, \| * \|_{\infty})$.

Theorems on denseness

By using the McShane-Whitney extension theorem, we obatain:

Theorem 32 ([I])

Let X be a metrizable compact finite-dimensional space. Let Q be the set of all metrics $d \in \mathrm{M}(X)$ for which (X, d) is isometrically embeddable into a Euclidean space equipped with ℓ^{∞}-norm. Then the set Q is dense in $\mathrm{M}(X)$.

By using Theorem 32 we obtain:

Theorem 33 ([I])

Let X be a metrizable topological space. Then X is finite-dimensional and compact if and only if the set of all doubling metrics in $\mathrm{M}(X)$ is dense F_{σ} in $\mathrm{M}(X)$.

This theorem characterize the finite-dimensionality and compactness.

Theorems on denseness

The following theorem characterize the zero-dimensionality and compactness by the denseness of the set of all UD metrics.

Theorem 34 ([I])

Let X be a metrizable topological space. Then X is 0-dimensional and compact if and only if the set of all uniformly disconnected metrics in $\mathrm{M}(X)$ is dense F_σ in $\mathrm{M}(X)$.

Theorems on denseness

A range set S has countable coinitiality if there exists $\{r_i\}_{i \in \mathbb{N}}$ in S such that $r_i \to 0$ as $i \to \infty$.
The following theorem characterize the compactness by the denseness of the set of all doubling metrics.
Note that all ultrametrizable space are zero-dimensional, then they are also finite-dimensional.

Theorem 35 ([I])

Let S be a range set with the countable coinitiality. Let X be an ultrametrizable space. Then X is compact if and only if the set of all doubling metrics in $\mathrm{UM}(X, S)$ is dense F_σ in $\mathrm{UM}(X, S)$.

Theorems on denseness

Recall that Γ stands for the middle third Cantor set.

Theorem 36 ([I])

The following statements hold true:

1. *The set of all uniformly perfect metrics in* $\mathrm{M}(\Gamma)$ *is dense* F_σ *in* $\mathrm{M}(\Gamma)$.
2. *The set of all non-uniformly perfect metrics in* $\mathrm{M}(\Gamma)$ *is dense* G_δ *in* $\mathrm{M}(\Gamma)$.

Definition 18

For a triple $(u, v, w) \in \{0, 1\}^3$*,*
a metric space (X, d) *has type* (u, v, w) *if*

$$T_D(X, d) = u, \quad T_{UD}(X, d) = v, \quad T_{UP}(X, d) = w.$$

For a triple $(u, v, w) \in \{0, 1\}^3$, we define $\mathcal{M}(u, v, w)$ by

$$\mathcal{M}(u, v, w) \\ = \{\mathcal{G}(X, d) \mid (X, d) \text{ is a Cantor metric space of type } (u, v, w)\}.$$

Theorem 37 ([I])

For every $(u, v, w) \in \{0, 1\}$ *except* $(1, 1, 1)$*, we have*

$$\mathrm{Card}(\mathcal{M}(u, v, w)) = 2^{\aleph_0},$$

where Card *denotes the cardinality.*

Theorems on QS invariant propertises

For every $(u, v, w) \in \{0, 1\}^3$, we put

$$\mathrm{E}(u, v, w) = \{\, d \in \mathrm{M}(\Gamma) \mid (\Gamma, d) \text{ is of type } (u, v, w) \,\}.$$

The following theorem is a cross point of my paper [0] and my paper [1] (and [2]).

Theorem 38 ([I])

The following three statements hold true:

(1) *The set* $\mathrm{E}(1, 1, 1)$ *is a dense* F_σ *set of* $\mathrm{M}(\Gamma)$.

(2) *The set* $\mathrm{E}(0, 0, 0)$ *is a dense* G_δ *subset of* $\mathrm{M}(\Gamma)$.

(3) *For every* $(u, v, w) \in \{0, 1\}^3$ *except* $(1, 1, 1)$ *and* $(0, 0, 0)$, *the set* $\mathrm{E}(u, v, w)$ *is a dense* $G_{\delta\sigma}$ *and* $F_{\sigma\delta}$ *subset of* $\mathrm{M}(\Gamma)$.

Thank you very much for your attention.