Branching geodesics of the Gromov–Hausdorff distance

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In this talk, I construct branching and confluent geodesics in the Gromov–Hausdorff space.

The Gromov–Hausdorff space

Definition 1

For a metric space (Z, h), and for subsets A, B of (Z, h), we denote by $\mathcal{H}(A, B; Z, h)$ the Hausdorff distance of A and B in Z. For metric spaces (X, d) and (Y, e), the **Gromov–Hausdorff distance** $\mathcal{GH}((X, d), (Y, e))$ between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z, h)$, where (Z, h) is a metric space, and $i : (X, d) \to (Z, h)$ and $j : (Y, e) \to (Z, h)$ are isometric embeddings.

We denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the **Gromov–Hausdorff space**.

Motivations

Theorem (David–Semmes, 1997)

If a compact metric space is doubling, uniformly disconnected, and uniformly perfect, then it is quasi-symmetrically equivalent to the Cantor set Γ equipped with the Euclidean metric.

Theorem (Ivanov–Nikolaeva–Tuzhilin (2016), Chowdhury–Mémoli (2018))

The space $(\mathcal{M}, \mathcal{GH})$ is a geodesic space, i.e., for all (X_0, d_0) and (X_1, d_1), there exists compact metric spaces $\{(X_t, d_t)\}_{t \in [0,1]}$ such that $\mathcal{GH}((X_t, d_t), (X_s, d_s)) = |t - s| \cdot \mathcal{GH}((X_0, d_0), (X_1, d_1))$.

Chowdhury–Mémoli (2018) showed that there exists branching and confluent geodesics in the Gromov–Hausdorff space.

Definition 2

If a metric space (X, d) satisfies a property $\mathcal{P}$, then we write $T_{\mathcal{P}}(X, d) = 1$; otherwise, $T_{\mathcal{P}}(X, d) = 0$. For a triple $(u_1, u_2, u_3) \in \{0, 1\}^3$, we say that a metric space (X, d) is *of type* (u_1, u_2, u_3) if we have $T_{\mathcal{P}_k}(X, d) = u_k$ for all $k \in \{1, 2, 3\}$.

For every $(u, v, w) \in \{0, 1\}^3$, we put

$$\mathrm{E}(u, v, w) = \{\, d \in \mathrm{Met}(\Gamma) \mid (\Gamma, d) \text{ is of type } (u, v, w) \,\}.$$

Theorem A ([I])

The following three statements hold true:

1. *The set* $\mathrm{E}(1, 1, 1)$ *is a dense* F_σ *set of* $(\mathrm{Met}(\Gamma), \mathcal{D}_\Gamma)$.
2. *The set* $\mathrm{E}(0, 0, 0)$ *is a dense* G_δ *subset of* $(\mathrm{Met}(\Gamma), \mathcal{D}_\Gamma)$.
3. *For every* $(u, v, w) \in \{0, 1\}^3$ *except* $(1, 1, 1)$ *and* $(0, 0, 0)$, *the set* $\mathrm{E}(u, v, w)$ *is a dense* $G_{\delta\sigma}$ *and* $F_{\sigma\delta}$ *subset of* $(\mathrm{Met}(\Gamma), \mathcal{D}_\Gamma)$.

Main results

Theorems 1 and 2 are analogues of my theorem (Theorem A) on the topological distribution of the doubling property, the uniform disconnectedness, and the uniform perfectness in the spaces of metrics.
Let $\mathcal{D}$, $\mathcal{UD}$, and $\mathcal{UP}$ be the sets of all doubling metric spaces, all uniformly disconnected metric spaces, and all uniformly perfect metric spaces in $\mathcal{M}$, respectively.

Theorem 1

The sets $\mathcal{D}$, $\mathcal{UD}$, and $\mathcal{UP}$ are dense F_σ and meager in the Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$.

Let $\mathcal{Q}_1 = \mathcal{D}$, $\mathcal{Q}_2 = \mathcal{U}\mathcal{D}$, and $\mathcal{Q}_3 = \mathcal{U}\mathcal{P}$. For $k \in \{1, 2, 3\}$, and for $u \in \{0, 1, 2\}$, we define

$$\mathcal{E}_k(u) = \begin{cases} \mathcal{M} \setminus \mathcal{Q}_k & \text{if } u = 0; \\ \mathcal{Q}_k & \text{if } u = 1; \\ \mathcal{M} & \text{if } u = 2. \end{cases}$$

For $(u, v, w) \in \{0, 1, 2\}^3$, we also define

$$\mathcal{X}(u, v, w) = \mathcal{E}_1(u) \cap \mathcal{E}_2(v) \cap \mathcal{E}_3(w).$$

Theorem 2

Let $(u, v, w) \in \{0, 1, 2\}^3$. *Then the following statements hold true.*

1. *If* $\{u, v, w\} \setminus \{2\} = \{1\}$, *then the set* $\mathcal{X}(u, v, w)$ *is dense* F_{σ}.
2. *If* $\{u, v, w\} \setminus \{2\} = \{0\}$, *then the set* $\mathcal{X}(u, v, w)$ *is dense* G_{δ}.
3. *If* $\{u, v, w\} \setminus \{2\} = \{0, 1\}$, *then the set* $\mathcal{X}(u, v, w)$ *is dense* $F_{\sigma\delta}$ *and* $G_{\delta\sigma}$.

Branching geodesics

We denote by $\mathbf{Q}$ the Hilbert cube, i.e., $\mathbf{Q}$ is a topological space homeomorphic to the countable power of the unit interval $[0, 1]$.

Definition 3

Let A be a closed subset of $[0, 1]$. Let $(X, d), (Y, e) \in \mathcal{M}$. We say that a continuous map $F : [0, 1] \times \mathbf{Q} \to \mathcal{M}$ is an A-**branching bunch of geodesics from** (X, d) **to** (Y, e) if the following are satisfied:

1. for every $q \in \mathbf{Q}$, we have $F(0, q) = (X, d)$ and $F(1, q) = (Y, e)$;
2. for every $s \in A$ and for all $q, r \in \mathbf{Q}$ we have $F(s, q) = F(s, r)$;
3. for each $q \in \mathbf{Q}$, the map $F_q : [0, 1] \to \mathcal{M}$ defined by $F_q(s) = F(s, q)$ is a geodesic from (X, d) to (Y, e);
4. for all $(s, q), (t, r) \in ([0, 1] \setminus A) \times \mathbf{Q}$ with $(s, q) \neq (t, r)$, we have $F(s, q) \neq F(t, r)$.

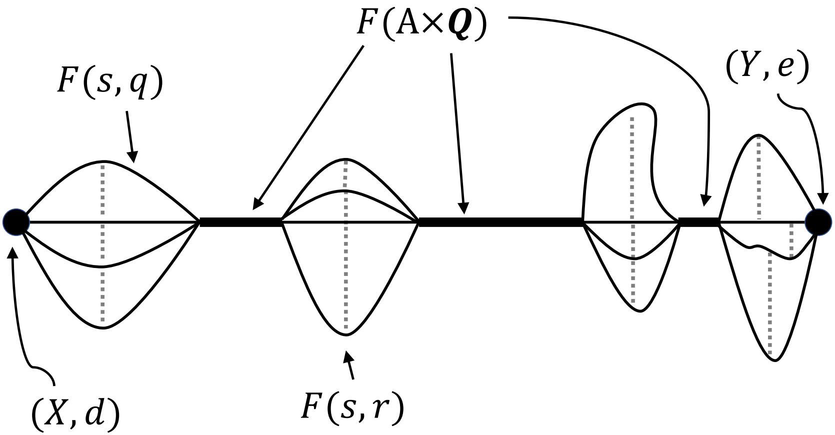

Figure: A-branching bunch of geodesics

The concept of A-branching bunch of geodesics represents branching and confluent geodesics in the Gromov–Hausdorff space. Our next results show that such geodesics actually exist.

Theorem 3

Let $(u, v, w) \in \{0, 1, 2\}^3$. Let $(X, d), (Y, e) \in \mathcal{X}(u, v, w)$. Let A be a closed subset of $[0, 1]$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathcal{X}(u, v, w)$ from (X, d) to (Y, e).

We say that a map $\mathcal{D} : \mathcal{M} \to [0, \infty]$ is a *dimensional function* if

1. for every $(X, d) \in \mathcal{M}$, and for every closed subset A of X, we have $\mathcal{D}(A, d) \leq \mathcal{D}(X, d)$;
2. for all $\epsilon \in (0, \infty)$, there exists a compact metric space (Y, e) with $\mathcal{D}(Y, e) = \infty$ and $\delta_e(Y) \leq \epsilon$.

For example, the covering dimension (topological dimension) dim, the Hausdorff dimension $\dim_H$, the packing dimension $\dim_P$, the lower box dimension $\underline{\dim}_B$, the upper box dimension $\overline{\dim}_B$, and the Assouad dimension $\dim_A$ are dimensional functions.

For a dimensional function $\mathcal{D}$, we denote by $\mathfrak{I}(\mathcal{D})$ the set of all compact metric spaces $(X, d) \in \mathcal{M}$ with $\mathcal{D}(X, d) = \infty$. Note that $\mathfrak{I}(\mathcal{D}) \neq \emptyset$. The sets of infinite metric spaces contains A-branching bunch of geodesics.

Theorem 4

Let $\mathcal{D}$ be a dimensional function. Let $(X, d), (Y, e) \in \mathfrak{I}(\mathcal{D})$, and A a closed subset of $[0, 1]$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathfrak{I}(\mathcal{D})$ from (X, d) to (Y, e).

We denote by $\mathcal{C}$ the set of all compact metric spaces homeomorphic to the Cantor set.
The set $\mathcal{C}$ also contain A-branching bunch of geodesics.

Theorem 5

Let $(X, d), (Y, e) \in \mathcal{C}$, and A a closed subset of $[0, 1]$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathcal{C}$ from (X, d) to (Y, e).

Corollaries

As a corollary, we see that there exists continuum many geodesics between two points in the Gromov–Hausdorff space.

Theorem 6

Let $\mathcal{S}$ be any one of $\mathcal{X}(u, v, w)$ for some $(u, v, w) \in \{0, 1, 2\}^3$ or $\mathcal{I}(\mathcal{D})$ for some dimensional function $\mathcal{D}$ or $\mathcal{C}$. Then for all $(X, d), (Y, e) \in \mathcal{S}$ satisfying $\mathcal{GH}((X, d), (Y, e)) > 0$, there are exact continuum many geodesics from (X, d) to (Y, e) passing through $\mathcal{S}$.

Our construction of geodesics implies a topological embedding of the Hilbert cube into the Gromov–Hausdorff space.

Theorem 7

Let $\mathcal{S}$ be any one of $\mathcal{X}(u, v, w)$ for some $(u, v, w) \in \{0, 1, 2\}^3$ or $\mathcal{I}(\mathcal{D})$ for some dimensional function $\mathcal{D}$ or $\mathcal{C}$. Then for all $(X, d), (Y, e) \in \mathcal{S}$, there exists a topological embedding from the Hilbert cube $\mathbf{Q}$ into $\mathcal{S}$ whose image contains (X, d) and (Y, e). In particular, $\mathcal{S}$ is infinite-dimensional.

Theorem 8

Let $\mathcal{S}$ be any one of $\mathcal{X}(u, v, w)$ for some $(u, v, w) \in \{0, 1, 2\}^3$ or $\mathcal{I}(\mathcal{D})$ for some dimensional function $\mathcal{D}$ or $\mathcal{C}$. Then for every non-empty open subset O of $\mathcal{M}$, the set $\mathcal{S} \cap O$ is infinite-dimensional.

Thank you very much for your attention.