Dense subsets in spaces of metrics

Yoshito Ishiki (伊敷喜斗)

Photonics Control Technology Team,
RIKEN Center for Advanced Photonics.

MSJ Autumn Meeting 2022 at Hokkaido University
2022/September/15

In this talk, I investigated the topological distributions of the doubling property, the uniform disconnectedness, and the uniform perfectness in the spaces of metrics.

Space of Metrics

Definition 1

For a metrizable space X, we denote by $\mathrm{Met}(X)$ the set of all metrics which generate the same topology as the topology of X.

Definition 2

For a metrizable space X, we define a function $\mathcal{D}_X : \mathrm{Met}(X)^2 \to [0, \infty]$ by

$$\mathcal{D}_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|.$$

Remark

The function $\mathcal{D}_X$ is a metric on $\mathrm{Met}(X)$ valued in $[0, \infty]$.

The doubling property

For a metic space (X, d) and for a subset A of X, we set

$$\alpha_d(A) = \inf\{\, d(x, y) \mid x, y \in A \text{ and } x \neq y \,\}.$$

A metric space (X, d) is said to be doubling if there exist $C \in (0, \infty)$ and $\beta \in (0, \infty)$ such that for every finite subset A of X we have

$$\mathrm{card}(A) \leq C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^{\beta},$$

where δ_d stands for the diameter.

Remark

A metric space satisfies the doubling property if and only if it has finite Assouad dimension.

The uniform disconnectedness

A metic space (X, d) is said to be uniformly disconnected if there exists $\delta \in (0, 1)$ such that for every finite sequence $\{z_i\}_{i=1}^{N}$ in X the inequality

$$d(z_i, z_{i+1}) \leq \delta d(z_1, z_N)$$

implies

$$z_1 = z_2 = \cdots = z_N.$$

Remark

A metric space is uniformly disconnected if and only if it is bi-Lipschitz homeomorphic to an ultrametric space.

Ultrametric: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

The uniform perfectness

A metric space (X, d) is said to be uniformly perfect if there exists $c \in (0, 1)$ such that for every $x \in X$, and for every $r \in (0, \delta_d(X))$, there exists $y \in X$ with $c \cdot r \leq d(x, y) \leq r$. Equivalently, $B(x, r) \setminus U(x, c \cdot r) \neq \emptyset$.

Remark

A subspace of an uniformly perfect space is not necessarily uniformly perfect.

Motivations

David–Semmes characterized the quasi-symmetric equivalence class of the standard Cantor set using the doubling property, the uniform disconnectedness, and the uniform perfectness.

Theorem (David–Semmes, 1997)

If a compact metric space is doubling, uniformly disconnected, and uniformly perfect, then it is quasi-symmetrically equivalent to the Cantor set Γ *equipped with the Euclidean metric.*

Motivations

I investigated the topological distribution of the negations of the doubling property and the uniform disconnectedness.

Theorem ([I], 2020)

Let X be a non-discrete metrizable space. Then the following sets are dense G_δ in the space $\mathrm{Met}(X)$ of metrics.

(1) *the set of all metric $d \in \mathrm{Met}(X)$ for which (X, d) is not doubling;*

(2) *the set of all $d \in \mathrm{Met}(X)$ for which (X, d) is not uniformly disconnected.*

Niemytzki–Tychonoff characterized the compactness of metric spaces.

Theorem (Niemytzki–Tychonoff, 1928)

Let X be a metric space. Then, X is compact if and only if every $d \in \mathrm{Met}(X)$ is a complete metric.

Main results

I proved the characterization of the finite-dimensionality using the doubling property. This result presents the topological distribution of the doubling property.

Theorem 1 ([I])

Let X be a metrizable compact finite-dimensional space. Let Q be the set of all metrics $d \in \mathrm{Met}(X)$ for which (X, d) is isometrically embeddable into a Euclidean space equipped with ℓ^{∞}-norm. Then the set Q is dense in $(\mathrm{Met}(X), \mathcal{D}_X)$.

Theorem 2 ([I])

Let X be a metrizable topological space. Then X is finite-dimensional and compact if and only if the set of all doubling metrics in $\mathrm{Met}(X)$ is dense F_{σ} in $(\mathrm{Met}(X), \mathcal{D}_X)$.

I proved the characterization of the 0-dimensionality using the uniform disconnectedness. This result presents the topological distribution of the uniform disconnectedness.

Theorem 3 ([I])

Let X be a metrizable space. Then X is 0-dimensional and compact if and only if the set of all uniformly disconnected metrics in $\mathrm{Met}(X)$ is dense F_σ in $(\mathrm{Met}(X), \mathcal{D}_X)$.

Let Γ denote the middle-third Cantor set.
I determined the topological distribution of the uniform perfectness in the space of metrics on Γ.

Theorem 4 ([I])

The following statements hold true:

1. *The set of all uniformly perfect metrics in* $\mathrm{Met}(\Gamma)$ *is dense* F_σ *in* $(\mathrm{Met}(\Gamma), \mathcal{D}_\Gamma)$.
2. *The set of all non-uniformly perfect metrics in* $\mathrm{Met}(\Gamma)$ *is a dense* G_δ *set in* $(\mathrm{Met}(\Gamma), \mathcal{D}_\Gamma)$.

In the following theorem (Theorem 5 in the next page), I determine the topological distributions of the doubling property, the uniform disconnectedness, and the uniform perfectness, and their negations, in the space of metrics on the Cantor set.

Definition 3

If a metric space (X, d) satisfies a property $\mathcal{P}$, then we write $T_{\mathcal{P}}(X, d) = 1$; otherwise, $T_{\mathcal{P}}(X, d) = 0$. For a triple $(u_1, u_2, u_3) \in \{0, 1\}^3$, we say that a metric space (X, d) is *of type* (u_1, u_2, u_3) if we have $T_{\mathcal{P}_k}(X, d) = u_k$ for all $k \in \{1, 2, 3\}$.

For every $(u, v, w) \in \{0, 1\}^3$, we put

$$\mathrm{E}(u, v, w) = \{\, d \in \mathrm{Met}(\Gamma) \mid (\Gamma, d) \text{ is of type } (u, v, w) \,\}.$$

Theorem ([I], 2019)

If $(u, v, w) \neq (1, 1, 1)$, *then the cardinality of the quasi-symmetrically equivalent classes of* $\mathrm{E}(u, v, w)$ *is equal to* $2^{\aleph_0}$.

Theorem 5 ([I])

The following three statements hold true:

1. *The set* $\mathrm{E}(1,1,1)$ *is a dense* F_{σ} *set of* $(\mathrm{Met}(\Gamma), \mathcal{D}_{\Gamma})$.
2. *The set* $\mathrm{E}(0,0,0)$ *is a dense* G_{δ} *subset of* $(\mathrm{Met}(\Gamma), \mathcal{D}_{\Gamma})$.
3. *For every* $(u, v, w) \in \{0,1\}^3$ *except* $(1,1,1)$ *and* $(0,0,0)$, *the set* $\mathrm{E}(u, v, w)$ *is a dense* $G_{\delta\sigma}$ *and* $F_{\sigma\delta}$ *subset of* $(\mathrm{Met}(\Gamma), \mathcal{D}_{\Gamma})$.

Thank you very much for your attention.