Fractal dimensions and topological embeddings of the Hilbert cube into the Gromov–Hausdorff space

Yoshito Ishiki (伊敷喜斗)

RIKEN, Photonics Control Technology Team

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In this talk, I construct topological embeddings from all compact metrizable spaces into subsets of Gromov–Hausdorff space.

The Gromov–Hausdorff space

The Gromov–Hausdorff space is a moduli space of all compact metric spaces, and it can be considered as a "universal" generalization of the Hausdorff hyperspace equipped with the Hausdorff distance.

Definition 1

For a metric space (Z, h), and for subsets A, B of (Z, h), we denote by $\mathcal{H}(A, B; Z, h)$ the Hausdorff distance of A and B in Z. For metric spaces (X, d) and (Y, e), the **Gromov–Hausdorff distance** $\mathcal{GH}((X, d), (Y, e))$ between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z, h)$, where (Z, h) is a metric space, and $i : (X, d) \to (Z, h)$ and $j : (Y, e) \to (Z, h)$ are isometric embeddings.
We denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the **Gromov–Hausdorff space**.

The Gromov–Hausdorff space

- $\mathcal{GH}$ is actually a metric on $\mathcal{M}$.

Theorem A (Ivanov–Nikolaeva–Tuzhilin (2016), Chowdhury–Mémoli (2018))

The space $(\mathcal{M}, \mathcal{GH})$ is a geodesic space, i.e., for all (X_0, d_0) and (X_1, d_1), there exists compact metric spaces $\{(X_t, d_t)\}_{t \in [0,1]}$ such that $\mathcal{GH}((X_t, d_t), (X_s, d_s)) = |t - s| \cdot \mathcal{GH}((X_0, d_0), (X_1, d_1))$.

Theorem B ([I])

For all distinct $(X, d), (Y, e) \in \mathcal{M}$, there exist $2^{\aleph_0}$ many geodesics between (X, d) and (Y, e).

Motivation–Embeddings

Question

Can all separable metric spaces be isometrically embedded into the Gromov–Hausdorff space?

- The case of finite metic spaces were positively solved. (Ivanov–Tuzhilin, 2019).
- Even the case of compact metric spaces are not known yet.
- Other cases are not known.
- Ultrametric analogues of this question were positively perfectly solved. (Wan, 2021).
- It is know that the identity map of $\mathcal{M}$ is an only isometric automorphism on $(\mathcal{M}, \mathcal{GH})$. In particular, it is not isometric to the Urysohn universal space (Ivanov–Tuzhilin, 2019).
- As far as the speaker knows, it is not known the homeomorphism type of $\mathcal{M}$.

As a consequence of the speaker's theorem on the existence of branching geodesics in the Gromov–Hausdorff space, the speaker proved:
Let $\mathbf{Q} = [0,1]^{\aleph_0}$

Theorem C ([I])

Let $\mathcal{S}$ be any one of the following subsets of $\mathcal{M}$

1. $\mathcal{M}$
2. the set of all compact metric spaces with infinite topological dimension.
3. the set of all compact metric spaces homeomorphic to the Cantor set.

Then for all (X, d), (Y, e) in $\mathcal{S}$, there exists a topological embedding $I : \mathbf{Q} \to \mathcal{S}$ such that the image of I contains (X, d) and (Y, e). In particular, we see that the set $\mathcal{S}$ is infinite-dimensional.

The speaker want to know the size (dimension) of subsets of $\mathcal{M}$.

Part1: Main theorems

1. $\dim_{\mathrm{T}} X$: the topological dimension,
2. $\dim_{\mathrm{H}}(X, d)$: the Hausdorff dimension,
3. $\dim_{\mathrm{P}}(X, d)$: the packing dimension.
4. $\overline{\dim}_{\mathrm{B}}(X, d)$: the upper box dimension,
5. $\dim_{\mathrm{A}}(X, d)$: the Assouad dimension.

The topological dimension takes values in $\mathbb{Z}_{\geq -1} \cup \{\infty\}$, and the other four dimensions take values in $[0, \infty]$. For every bounded metric space (X, d), we have the following basic inequalities:

$$\dim_{\mathrm{T}} X \leq \dim_{\mathrm{H}}(X, d) \leq \dim_{\mathrm{P}}(X, d) \leq \overline{\dim}_{\mathrm{B}}(X, d) \leq \dim_{\mathrm{A}}(X, d).$$

We denote by $\mathcal{L}$ the set of all $(a_1, a_2, a_3, a_4) \in [0, \infty]^4$ satisfying

$$a_1 \leq a_2 \leq a_3 \leq a_4.$$

We also denote by $\mathcal{R}$ the set of all $(l, a_1, a_2, a_3, a_4) \in (\mathbb{Z}_{\geq 0} \cup \{\infty\}) \times [0, \infty]^4$ satisfying $l \leq a_1 \leq a_2 \leq a_3 \leq a_4$.
A metric d on X is said to be an *ultrametric* if it satisfies $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

Theorem 1 ([I])

For every $(a_1, a_2, a_3, a_4) \in \mathcal{L}$, there exists a Cantor ultrametric space (X, d) such that

$$\dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

By Theorem 1, we obtain the following :

Theorem 2 ([I])

For every $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$*, there exists a compact metric space* (X, d) *such that*

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

For $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$, we denote by $\mathcal{D}(l, a_1, a_2, a_3, a_4)$ the set of all compact metric spaces in $\mathcal{M}$ satisfying

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

A metric d on X is said to be an *ultrametric* if it satisfies $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.
We denote by $\mathcal{U}$ the set of all compact ultrametric spaces in $\mathcal{M}$.

Theorem D (Mémoli–Smith–Wan, 2019)

The set $\mathcal{U}$ is a geodesic subspace of $(\mathcal{M}, \mathcal{GH})$.

- $\mathcal{D}(l, a_1, a_2, a_3, a_4)$: the set of all compact metric spaces in $\mathcal{M}$ satisfying

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

- $\mathcal{U}$: the set of all compact **ultrametric** spaces in $\mathcal{M}$.
- $[0, 1]^{\aleph_0}$: A Hilbert cube.

Theorem 3 ([I])

Assume that $\mathcal{S} = \mathcal{D}(l, a_1, a_2, a_3, a_4)$ for some numbers $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$ or $\mathcal{S} = \mathcal{U}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let H be a Hilbert cube, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be compact metric spaces in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j. Then there exists a topological embedding $\Phi : H \to \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$ for all $i \in \{1, \ldots, n+1\}$.

Remark

Since every compact metric space can be embedded into a Hilbert cube, Then Theorem 3 is even valid when H is an arbitrary compact metric space.

Remark

This talk is based on my paper "Fractal dimensions and topological embeddings of simplexes into the Gromov–Hausdorff space".
In my paper, the speaker only prove Theorem 3 in the case where H is a simplex. When writing the paper, the speaker noticed that Theorem 3 is still valid in the case where H is any compact metric space.

Thank you very much for your attention.