Branching and confluent geodesics in the Gromov–Hausdorff space

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In this talk, I construct branching and confluent geodesics in the Gromov–Hausdorff space.
This talk is based on

- Y. Ishiki, *Branching geodesics of the Gromov–Hausdorff distance*, arXiv:2108.06970, to appear in Analysis and Geometry in Metric Spaces.

Part 1

I first explain a method of constructing branching geodesics in the Gromov–Hausdorff space.

The Gromov–Hausdorff space

Definition 1

For a metric space (Z, h), and for subsets A, B of (Z, h), we denote by $\mathcal{H}(A, B; Z)$ the Hausdorff distance of A and B in (Z, h). For metric spaces (X, d) and (Y, e), the **Gromov–Hausdorff distance** $\mathcal{GH}((X, d), (Y, e))$ between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z)$, where (Z, h) is a metric space, and $i : (X, d) \to (Z, h)$ and $j : (Y, e) \to (Z, h)$ are isometric embeddings.

We denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the **Gromov–Hausdorff space**.

Motivations

Theorem A (Ivanov–Nikolaeva–Tuzhilin (2016), Chowdhury–Mémoli (2018))

The space $(\mathcal{M}, \mathcal{GH})$ *is a geodesic space, i.e., for all* (X_0, d_0) *and* (X_1, d_1)*, there exists compact metric spaces* $\{(X_t, d_t)\}_{t\in[0,1]}$ *such that* $\mathcal{GH}((X_t, d_t), (X_s, d_s)) = |t - s| \cdot \mathcal{GH}((X_0, d_0), (X_1, d_1))$.

Chowdhury–Mémoli (2018) showed that there exists branching and confluent geodesics in the Gromov–Hausdorff space.
Mémoli–Wan (2021) constructed countably many geodesics between given two compact metric spaces.

Branching and confluent geodesics

We denote by $\mathbf{Q}$ the Hilbert cube, i.e., $\mathbf{Q}$ is the countable power of the unit interval $[0, 1]$.

Definition 2 ([I])

Let A be a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Let $(X, d), (Y, e) \in \mathcal{M}$. We say that a continuous map $F : [0, 1] \times \mathbf{Q} \to \mathcal{M}$ is an A-**branching bunch of geodesics from** (X, d) **to** (Y, e) if the following are satisfied:

1. for every $q \in \mathbf{Q}$, we have $F(0, q) = (X, d)$ and $F(1, q) = (Y, e)$;
2. for every $s \in A$ and for all $q, r \in \mathbf{Q}$ we have $F(s, q) = F(s, r)$;
3. for each $q \in \mathbf{Q}$, the map $F_q : [0, 1] \to \mathcal{M}$ defined by $F_q(s) = F(s, q)$ is a geodesic from (X, d) to (Y, e);
4. for all $(s, q), (t, r) \in ([0, 1] \setminus A) \times \mathbf{Q}$ with $(s, q) \neq (t, r)$, we have $F(s, q) \neq F(t, r)$.

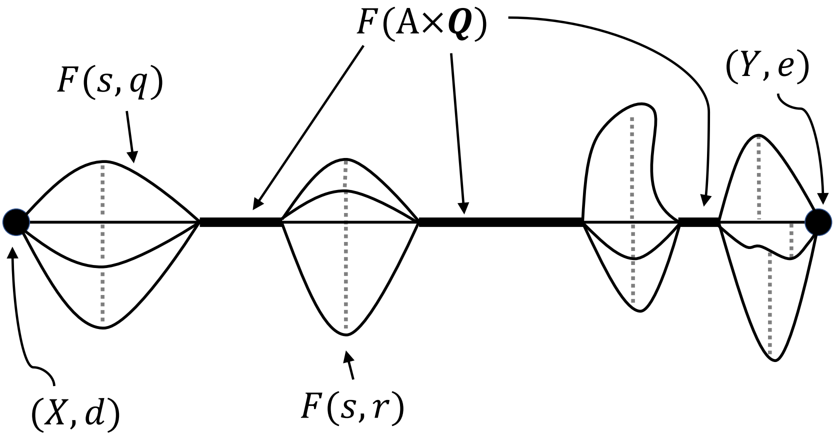


Figure: A-branching bunch of geodesics

Construction of branching bunch of geodesics

Theorem 1 ([I])

Let $(X, d), (Y, e) \in \mathcal{M}$, and A a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathcal{M}$ from (X, d) to (Y, e).

Sketch of Proof

Let $x \vee y = \max\{x, y\}$.

Lemma 1

Let $x, y, u, v \in \mathbb{R}$. Then, we have

$$|x \vee y - u \vee v| \leq |x - u| \vee |y - v|.$$

Lemma 2

Let $\gamma : [0,1] \to X$ be a curve. If for every $s, t \in [0,1]$, we have $d(\gamma(s), \gamma(t)) \leq |s - t| \cdot d(\gamma(0), \gamma(1))$, then γ is a geodesic.

1. Take a geodesic γ
2. Expand γ. (Take a product with a Cantor set)
3. Construct "identifiers"
4. Add "identifiers" to the modified γ.
5. We obtain an A-branching bunch of geodesics.

Constructing "Expansion" of a geodesic γ

Being a geodesic is invariant under ℓ^∞-product.
$(d \times_\infty e)((x, y), (v, w)) = d(x, v) \vee e(y, w)$.

Proposition 1 ([I])

Let (X_0, d_0) and (X_1, d_1) be compact metric spaces with $\mathcal{GH}((X_0, d_0), (X_1, d_1)) > 0$. Let $\gamma : [0, 1] \to \mathcal{M}$ be a geodesic from (X_0, d_0) to (X_1, d_1). Let (C, h) be in $\mathcal{M}$, and $L \in (0, \infty)$ satisfy $L\delta_h(C) \leq 2\,\mathcal{GH}((X_0, d_0), (X_1, d_1))$. Let A be a closed subset of $[0, 1]$ with $0, 1 \in A$, and let $\zeta : [0, 1] \to [0, \infty)$ be an L-Lipschitz map with $\zeta^{-1}(0) = A$. Put $\gamma(s) = (X_s, d_s)$. We define a map $\gamma \times_\infty C : [0, 1] \to \mathcal{M}$ by

$$(\gamma \times_\infty C)(s) = \begin{cases} \gamma(s) & \text{if } s \in A; \\ (X_s \times C, d_s \times_\infty (\zeta(s) \cdot h)) & \text{if } s \in [0, 1] \setminus A. \end{cases}$$

Then $(\gamma \times_\infty C)$ is a geodesic from (X_0, d_0) to (X_1, d_1).

A metric space is said to be **perfect** if it has no isolated points.

Corollary 1

Let (X_0, d_0) and (X_1, d_1) be compact metric spaces with $\mathcal{GH}((X_0, d_0), (X_1, d_1)) > 0$. Let $\gamma : [0, 1] \to \mathcal{M}$ be a geodesic from (X_0, d_0) to (X_1, d_1). Then, there exists a geodesic γ' such that γ and γ' coincides with each other on A, and if $s \notin A$ each $\gamma'(s)$ is a perfect metric space.

Constructing "identifiers"

Proposition 2 ([I])

There exists a family $\{(U_q, e_q)\}_{q\in\mathbf{Q}}$ *such that*

1. $U : \mathbf{Q} \to \mathcal{M}$ *defined by* $U(q) = (U_q, e_q)$ *is continuous;*
2. *Each* (U_q, e_q) *is countable and has only one accumulation point;*
3. *Let* $K, L > 0$ *and* $q, r \in \mathbf{Q}$. *If* $(U_q, K \cdot e_q)$ *and* $(U_r, L \cdot e_r)$ *are isometric to each other, then* $q = r$.

Adding “identifiers”

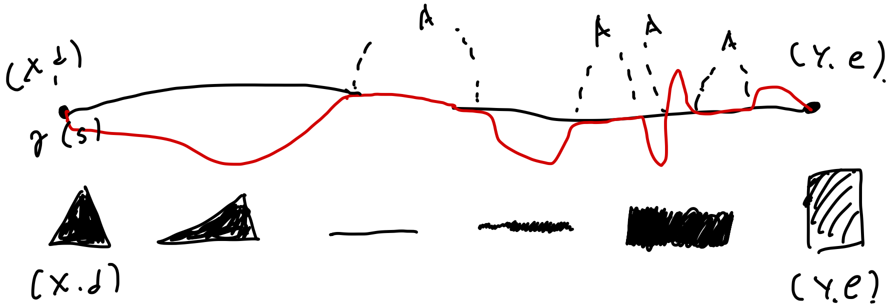
Being a geodesic is invariant under wedge sum.

Proposition 3 ([I])

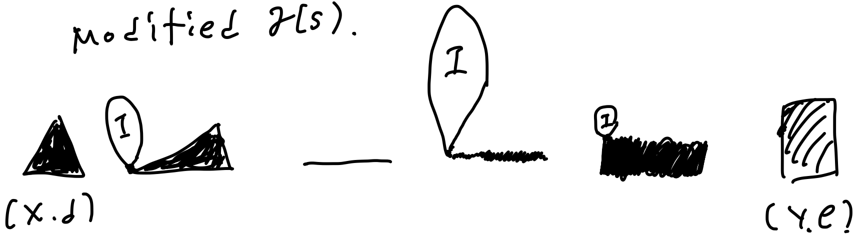
Let (X_0, d_0) *and* (X_1, d_1) *be compact metric spaces with* $\mathcal{GH}((X_0, d_0), (X_1, d_1)) > 0$. *Let* $\gamma : [0, 1] \to \mathcal{M}$ *be a geodesic from* (X_0, d_0) *to* (X_1, d_1). *Let* $(I, m) \in \mathcal{M}$, *and let* $L \in (0, \infty)$ *satisfy* $L\delta_e(I) \leq 2\,\mathcal{GH}((X_0, d_0), (X_1, d_1))$. *Let* A *be a closed subset of* $[0, 1]$ *with* $\{0, 1\} \subset A$, *and let* $\zeta : [0, 1] \to [0, \infty)$ *be an* L*-Lipschitz map with* $\zeta^{-1}(0) = A$. *We assume that there exists a set* R *such that for all* $s \in (0, 1)$ *we can represent* $\gamma(s) = (R, d_s)$. *Let* $o \in R$, *and* $\omega \in Y$. *We define* $Z = R \times \{\omega\} \cup \{o\} \times I$, *and define a map* $\gamma * I : [0, 1] \to \mathcal{M}$ *by*

$$(\gamma * I)(s) = \begin{cases} \gamma(s) & \text{if } s \in A; \\ (Z, d_s \times_\infty (\zeta(s) \cdot m)) & \text{if } s \in [0, 1] \setminus A. \end{cases}$$

Then, $\gamma * I$ *is a geodesic from* (X_0, d_0) *to* (X_1, d_1).



modified $\gamma(s)$.



Constructing bunch of geodesics

1. Take a geodesic γ
2. Expand γ. (Take a product with a Cantor set)
3. Construct "identifiers"
4. Add "identifiers" to the modified γ.
5. We obtain an A-branching bunch of geodesics.

Construction

Fix A. Take a geodesic γ from (X, d) to (Y, e).
Let (C, d_C) be a Cantor set. Put $\gamma' = (\gamma \times_\infty C)$.
Take identifiers (U_q, e_q). Define $F : [0, 1] \times \mathbf{Q} \to \mathcal{M}$ by
$F(s, q) = (\gamma' * U_q)(s)$.
Then F is an A-branching bunch of geodesics from (X, d) to (Y, e).

1. Take a geodesic γ
2. Expand γ. (Take a product with a Cantor set)
3. Construct "identifiers"
4. Add "identifiers" to the modified γ.
5. We obtain an A-branching bunch of geodesics.

Part 2

I next explain theorems in Abstract of my talk.

The doubling property

For a metic space (X, d) and for a subset A of X, we set

$$\alpha_d(A) = \inf\{\, d(x, y) \mid x, y \in A \text{ and } x \neq y \,\}.$$

A metric space (X, d) is said to be **doubling** if there exist $C \in (0, \infty)$ and $\alpha \in (0, \infty)$ such that for every finite subset A of X we have

$$\mathrm{card}(A) \leq C \left(\frac{\delta_d(A)}{\alpha_d(A)} \right)^{\alpha},$$

where δ_d stands for the diameter.

Remark

A metric space satisfies the doubling property if and only if it has finite Assouad dimension.

The uniform disconnectedness

Definition 3

A metric d is said to be an **ultrametric** if $d(x, y) \le d(x, z) \vee d(z, y)$.

Definition 4

A map $f : (x, d) \to (Y, e)$ is said to be a **bi-Lipschitz** if there exists $L \ge 1$ such that $L^{-1}d(x, y) \le d(f(x), f(y)) \le Ld(x, y)$.

A metic space (X, d) is said to be **uniformly disconnected** if it is bi-Lipschitz homeomorphic to an ultrametric space. Namely, there exist an ultrametric e on X and $L \ge 1$ such that
$L^{-1}d(x, y) \le e(x, y) \le Ld(x, y)$.

The uniform perfectness

A metric space (X, d) is said to be **uniformly perfect** if there exists $c \in (0, 1)$ such that for every $x \in X$, and for every $r \in (0, \delta_d(X))$, there exists $y \in X$ with $c \cdot r \leq d(x, y) \leq r$. Equivalently, $B(x, r) \setminus U(x, c \cdot r) \neq \emptyset$.

Remark

A subspace of an uniformly perfect space is not necessarily uniformly perfect.

Let $\mathcal{Q}_1$ be the set of all doubling spaces in $\mathcal{M}$, $\mathcal{Q}_2$ be the set of all uniformly disconnected spaces in $\mathcal{M}$, and $\mathcal{Q}_3$ be the set of all uniformly perfect spaces in $\mathcal{M}$. For $k \in \{1, 2, 3\}$, and for $u \in \{0, 1, 2\}$, we define

$$\mathcal{E}_k(u) = \begin{cases} \mathcal{M} \setminus \mathcal{Q}_k & \text{if } u = 0; \\ \mathcal{Q}_k & \text{if } u = 1; \\ \mathcal{M} & \text{if } u = 2. \end{cases}$$

For $(u, v, w) \in \{0, 1, 2\}^3$, we also define

$$\mathcal{X}(u, v, w) = \mathcal{E}_1(u) \cap \mathcal{E}_2(v) \cap \mathcal{E}_3(w).$$

Motivations

Theorem B (David–Semmes, 1997)

If a compact metric space is doubling, uniformly disconnected, and uniformly perfect, then it is quasi-symmetrically equivalent to the Cantor set Γ equipped with the Euclidean metric.

Quasi-symmetric maps

Definition 5

Let (X, d) and (Y, e) be metric spaces. A homeomorphism $f : X \to Y$ is said to be **quasi-symmetric** if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all $x, y, z \in X$ and for every $t \in [0, \infty)$ the inequality $d(x, y) \leq td(x, z)$ implies the inequality $e(f(x), f(y)) \leq \eta(t)e(f(x), f(z))$.

For example, all bi-Lipschitz homeomorphisms are quasi-symmetric. Note that the doubling property, the uniform disconnectedness, and the uniform perfectness are invariant under quasi-symmetric maps.

To simplify our description, the symbols $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ stand for the doubling property, the uniform disconnectedness, and the uniform perfectness, respectively.

Definition 6 ([I, 2018])

If a metric space (X, d) satisfies a property $\mathcal{P}$, then we write $T_{\mathcal{P}}(X, d) = 1$; otherwise, $T_{\mathcal{P}}(X, d) = 0$. For a triple $(u_1, u_2, u_3) \in \{0, 1\}^3$, we say that a metric space (X, d) is *of type* (u_1, u_2, u_3) if we have $T_{\mathcal{P}_k}(X, d) = u_k$ for all $k \in \{1, 2, 3\}$.

Theorem C ([I, 2018])

For all $(u_1, u_2, u_3) \neq (1, 1, 1)$*, there exists a Cantor metric space of type* (u_1, u_2, u_3)*. Moreover, the set of quasi-symmetric equivalent classes of Cantor metric space of type* (u_1, u_2, u_3) *are equal to* $2^{\aleph_0}$*.*

$$\mathrm{card}(\{\, [X, d]_{qs} \mid (X, d) \text{ is Cantor metric space of type } (u_1, u_2, u_3)\ \}) = 2^{\aleph_0}$$

For a metrizable space X, we denote by $\mathrm{M}(X)$ the set of all metrics on X that generate the same topology on X. Define a metric $\mathcal{D}_X : \mathrm{M}(X)^2 \to [0, \infty]$ by $\mathcal{D}_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|$.
Let Γ be the Cantor set.
For every $(u, v, w) \in \{0, 1\}^3$, we put

$$\mathrm{E}(u, v, w) = \{\, d \in \mathrm{M}(\Gamma) \mid (\Gamma, d) \text{ is of type } (u, v, w) \,\}.$$

Theorem D ([I, 2021])

The following three statements hold true:

1. *The set* $\mathrm{E}(1, 1, 1)$ *is a dense* F_σ *set of* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.
2. *The set* $\mathrm{E}(0, 0, 0)$ *is a dense* G_δ *subset of* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.
3. *For every* $(u, v, w) \in \{0, 1\}^3$ *except* $(1, 1, 1)$ *and* $(0, 0, 0)$, *the set* $\mathrm{E}(u, v, w)$ *is a dense* $G_{\delta\sigma}$ *and* $F_{\sigma\delta}$ *subset of* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.

Main results

Theorems 2–4 are the existence of branching and confluent geodesics in subsets in the Gromov–Hausdorff space.

Theorem 5–7 are corollaries of the existence of branching and confluent geodesics.

Next I explain main results on the branching geodesics.
The concept of A-branching bunch of geodesics represents branching and confluent geodesics in the Gromov–Hausdorff space. Our next results show that such geodesics actually exist.

Theorem 2 ([I])

Let $(u, v, w) \in \{0, 1, 2\}^3$. Let $(X, d), (Y, e) \in \mathcal{X}(u, v, w)$. Let A be a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathcal{X}(u, v, w)$ from (X, d) to (Y, e).

We say that a map $\mathcal{D} : \mathcal{M} \to [0, \infty]$ is a *dimensional function* if

1. for every $(X, d) \in \mathcal{M}$, and for every closed subset A of X, we have $\mathcal{D}(A, d) \leq \mathcal{D}(X, d)$;
2. for all $\epsilon \in (0, \infty)$, there exists a compact metric space (Y, e) with $\mathcal{D}(Y, e) = \infty$ and $\delta_e(Y) \leq \epsilon$.

For example, the covering dimension (topological dimension) dim, the Hausdorff dimension $\dim_H$, the packing dimension $\dim_P$, the lower box dimension $\underline{\dim}_B$, the upper box dimension $\overline{\dim}_B$, and the Assouad dimension $\dim_A$ are dimensional functions.

For a dimensional function $\mathcal{D}$, we denote by $\mathcal{I}(\mathcal{D})$ the set of all compact metric spaces $(X, d) \in \mathcal{M}$ with $\mathcal{D}(X, d) = \infty$. Note that $\mathcal{I}(\mathcal{D}) \neq \emptyset$. The sets of infinite metric spaces contains A-branching bunch of geodesics.

Theorem 3 ([I])

Let $\mathcal{D}$ be a dimensional function. Let $(X, d), (Y, e) \in \mathcal{I}(\mathcal{D})$, and A a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathcal{I}(\mathcal{D})$ from (X, d) to (Y, e).

We denote by $\mathcal{C}$ the set of all compact metric spaces homeomorphic to the Cantor set.
The set $\mathcal{C}$ also contain A-branching bunch of geodesics.

Theorem 4 ([I])

Let $(X, d), (Y, e) \in \mathcal{C}$, and A a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Then there exists an A-branching bunch of geodesics $F : [0, 1] \times \mathbf{Q} \to \mathcal{C}$ from (X, d) to (Y, e).

Corollaries

Combining Theorems 2–4, We obtain

Theorem 5 ([I])

Let $\mathcal{S}$ be any one of

1. $\mathcal{X}(u, v, w)$ *for some* $(u, v, w) \in \{0, 1, 2\}^3$
2. $\mathcal{I}(\mathcal{D})$ *for some dimensional function* $\mathcal{D}$
3. $\mathcal{C}$.

Let $(X, d), (Y, e) \in \mathcal{S}$, *and* A *a closed subset of* $[0, 1]$ *with* $\{0, 1\} \subset A$. *Then there exists an* A*-branching bunch of geodesics* $F : [0, 1] \times \mathbf{Q} \to \mathcal{S}$ *from* (X, d) *to* (Y, e). *In particular,* $\mathcal{S}$ *is a geodesic space.*

As a corollary, we see that there exists continuum many geodesics between two points in the Gromov–Hausdorff space.

Theorem 6 ([I])

Let $\mathcal{S}$ be any one of

1. $\mathcal{X}(u, v, w)$ *for some* $(u, v, w) \in \{0, 1, 2\}^3$
2. $\mathcal{I}(\mathcal{D})$ *for some dimensional function* $\mathcal{D}$
3. $\mathcal{C}$.

Then for all $(X, d), (Y, e) \in \mathcal{S}$ *satisfying* $\mathcal{GH}((X, d), (Y, e)) > 0$, *there are exact continuum many geodesics from* (X, d) *to* (Y, e) *passing through* $\mathcal{S}$.

Our construction of geodesics implies a topological embedding of the Hilbert cube into the Gromov–Hausdorff space. (Apply Theorem 6 to $A = \{0, 1\}$.)

Theorem 7 ([I])

Let $\mathcal{S}$ be any one of

1. $\mathcal{X}(u, v, w)$ *for some* $(u, v, w) \in \{0, 1, 2\}^3$
2. $\mathcal{I}(\mathcal{D})$ *for some dimensional function* $\mathcal{D}$
3. $\mathcal{C}$.

Then for all (X, d), (Y, e) in $\mathcal{S}$, there exists a topological embedding $I : \mathbf{Q} \to \mathcal{S}$ such that tha image of I contains (X, d) and (Y, e). In particular, we see that the set $\mathcal{S}$ is infinite-dimensional.

Thank you very much for your attention.