Topological embeddings into the Gromov–Hausdorff space

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In this talk, I construct topological embeddings from all compact metrizable spaces into subsets of Gromov–Hausdorff space.

Part 0: Introduction

This talk is based on the following three papers:

(I1) Yoshito Ishiki, Fractal dimensions and topological embeddings of simplexes into the Gromov–Hausdorff space, arXiv:2110.01881.

(I2) Yoshito Ishiki, Continua in the Gromov–Hausdorff space, arXiv:2111.08199, to apper in Topology Appl.

(I3) Yoshito Ishiki, Metric trees in the Gromov–Hasudorff space, arXiv:2112.05345, to appear in Comment. Math. Univ. Carol.

Motivation–"Size" of geometric structures

For a topological space X, we denote by $\mathrm{Met}(X)$ the set of all metrics that generate the same topology of X, and we denote by $\mathcal{D}_X$ the supremum metric on $\mathrm{Met}(X)$.

Theorem A ([I], 2022)

Let X be a metrizable topological space. Then X is finite-dimensional and compact if and only if the set of all doubling metrics in $\mathrm{Met}(X)$ is dense F_σ in $(\mathrm{Met}(X), \mathcal{D}_X)$.

Theorem B ([I], 2022)

Let X be a metrizable space. Then X is 0-dimensional and compact if and only if the set of all uniformly disconnected metrics in $\mathrm{Met}(X)$ is dense F_σ in $(\mathrm{Met}(X), \mathcal{D}_X)$.

Let $\mathcal{D}$, $\mathcal{UD}$, and $\mathcal{UP}$ be the sets of all doubling metric spaces, all uniformly disconnected metric spaces, and all uniformly perfect metric spaces in $\mathcal{M}$, respectively.

Theorem C ([I], 2022)

The sets $\mathcal{D}$, $\mathcal{UD}$, and $\mathcal{UP}$ are dense F_σ and meager in the Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$.

The speaker want to know the size of geometric structures.

The Hausdorff distance

Before defining the Gromov–Hausdorff space, we introduce the Hausdorff distance.

Definition 1

Let (X, d) be a metric space. Let $A, B \subset X$. We define the Hausdorff distance $\mathcal{H}(A, B; X, d)$ by the infimum of all $r \in (0, \infty)$ such that $A \subset \bigcup_{b \in B} B(b, r)$ and $B \subset \bigcup_{a \in A} B(a, r)$.

The Gromov–Hausdorff space

The Gromov–Hausdorff space is a moduli space of all compact metric spaces, and it can be considered as a "universal" generalization of the Hausdorff hyperspace equipped with the Hausdorff distance.

Definition 2

For a metric space (Z, h), and for subsets A, B of (Z, h), we denote by $\mathcal{H}(A, B; Z, h)$ the Hausdorff distance of A and B in Z. For metric spaces (X, d) and (Y, e), the **Gromov–Hausdorff distance** $\mathcal{GH}((X, d), (Y, e))$ between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z)$, where (Z, h) is a metric space, and $i : (X, d) \to (Z, h)$ and $j : (Y, e) \to (Z, h)$ are isometric embeddings.

We denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the **Gromov–Hausdorff space**.

The Gromov–Hausdorff space

- $\mathcal{GH}$ is actually a metric on $\mathcal{M}$.

Theorem D (Ivanov–Nikolaeva–Tuzhilin (2016), Chowdhury–Mémoli (2018))

The space $(\mathcal{M}, \mathcal{GH})$ is a geodesic space, i.e., for all (X_0, d_0) and (X_1, d_1), there exists compact metric spaces $\{(X_t, d_t)\}_{t\in[0,1]}$ such that $\mathcal{GH}((X_t, d_t), (X_s, d_s)) = |t - s| \cdot \mathcal{GH}((X_0, d_0), (X_1, d_1))$.

Theorem E ([I])

For all distinct $(X, d), (Y, e) \in \mathcal{M}$, there exist $2^{\aleph_0}$ many geodesics between (X, d) and (Y, e).

Motivation–Embeddings

Question

Can all separable metric spaces be isometrically embedded into the Gromov–Hausdorff space?

- The case of finite metic spaces were positively solved. (Ivanov–Tuzhilin, 2019).
- Even the case of compact metric spaces are not known yet.
- Other cases are not known.
- Ultrametric analogues of this question were positively perfectly solved. (Wan, 2021).
- It is know that the identity map of $\mathcal{M}$ is an only isometric automorphism on $(\mathcal{M}, \mathcal{GH})$. In particular, it is not isometric to the Urysohn universal space (Ivanov–Tuzhilin, 2019).
- As far as the speaker knows, it is not known the homeomorphism type of $\mathcal{M}$.

As a consequence of the speaker's theorem on the existence of branching geodesics in the Gromov–Hausdorff space, the speaker proved:
Let $\mathbf{Q} = [0,1]^{\aleph_0}$

Theorem F ([I])

Let $\mathcal{S}$ be any one of the following subsets of $\mathcal{M}$

1. $\mathcal{M}$
2. the set of all compact metric spaces with infinite topological dimension.
3. the set of all compact metric spaces homeomorphic to the Cantor set.

Then for all (X, d), (Y, e) in $\mathcal{S}$, there exists a topological embedding $I : \mathbf{Q} \to \mathcal{S}$ such that the image of I contains (X, d) and (Y, e). In particular, we see that the set $\mathcal{S}$ is infinite-dimensional.

The speaker want to know the size (dimension) of subsets of $\mathcal{M}$.

Part1: Main theorems

1. $\dim_{\mathrm{T}} X$: the topological dimension,
2. $\dim_{\mathrm{T}} X$: the Hausdorff dimension,
3. $\dim_{\mathrm{P}}(X, d)$: the packing dimension.
4. $\overline{\dim}_{\mathrm{B}}(X, d)$: the upper box dimension,
5. $\dim_{\mathrm{A}}(X, d)$: the Assouad dimension.

The topological dimension takes values in $\mathbb{Z}_{\geq -1} \cup \{\infty\}$, and the other four dimensions take values in $[0, \infty]$. For every bounded metric space (X, d), we have the following basic inequalities:

$$\dim_{\mathrm{T}} X \leq \dim_{\mathrm{H}}(X, d) \leq \dim_{\mathrm{P}}(X, d) \leq \overline{\dim}_{\mathrm{B}}(X, d) \leq \dim_{\mathrm{A}}(X, d).$$

We denote by $\mathcal{L}$ the set of all $(a_1, a_2, a_3, a_4) \in [0, \infty]^4$ satisfying

$$a_1 \le a_2 \le a_3 \le a_4.$$

We also denote by $\mathcal{R}$ the set of all $(l, a_1, a_2, a_3, a_4) \in (\mathbb{Z}_{\ge 0} \cup \{\infty\}) \times [0, \infty]^4$ satisfying $l \le a_1 \le a_2 \le a_3 \le a_4$.

Theorem 1 ([I])

For every $(a_1, a_2, a_3, a_4) \in \mathcal{L}$*, there exists a Cantor ultrametric space* (X, d) *such that*

$$\dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

By Theorem 1, we obtain the following :

Theorem 2 ([I])

For every $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$, *there exists a compact metric space* (X, d) *such that*

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3,$$
$$\dim_{\mathrm{A}}(X, d) = a_4.$$

For $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$, we denote by $\mathcal{D}(l, a_1, a_2, a_3, a_4)$ the set of all compact metric spaces in $\mathcal{M}$ satisfying

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

A metric d on X is said to be an *ultrametric* if it satisfies $d(x, y) \leq d(x, z) \vee d(z, y)$.
We denote by $\mathcal{U}$ the set of all compact ultrametric spaces in $\mathcal{M}$.

Theorem G (Mémoli–Smith–Wan, 2019)

The set $\mathcal{U}$ is a geodesic subspace of $(\mathcal{M}, \mathcal{GH})$.

- $\mathcal{D}(l, a_1, a_2, a_3, a_4)$: the set of all compact metric spaces in $\mathcal{M}$ satisfying

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3,$$
$$\dim_{\mathrm{A}}(X, d) = a_4.$$

- $\mathcal{U}$: the set of all compact **ultrametric** spaces in $\mathcal{M}$.

Theorem 3 ([I])

Assume that $\mathcal{S} = \mathcal{D}(l, a_1, a_2, a_3, a_4)$ for some numbers $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$ or $\mathcal{S} = \mathcal{U}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let H be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be compact metric spaces in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j. Then there exists a topological embedding $\Phi : H \to \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$ for all $i \in \{1, \ldots, n+1\}$.

Remark

In the paper (I1), the speaker only prove Theorem 3 in the case where H is a simplex. When writing the paper (I2), the speaker noticed that Theorem 3 is still valid in the case where H is any compact metric space.

Let (X, d) be a metric space. In this paper, for $x, y \in X$, a map $\gamma : [0,1] \to X$ is said to be a *geodesic from x to y* if $\gamma(0) = x$ and $\gamma(1) = y$, and for all $s, t \in [0,1]$ we have $d(\gamma(s), \gamma(t)) = |s-t| \cdot d(x,y)$. Note that if there exists a curve from x to y whose length is $d(x,y)$, then there exists a geodesic from x to y. A metric space is said to be a *geodesic space* if for all two points, there exists a geodesic connecting them.
A geodesic space (X, d) is said to be a $\mathrm{CAT}(0)$ *space* if for all geodesic triangle $\triangle$ in (X, d) and for all $x, y \in \triangle$ we have $d(x,y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$, where $d_{\mathbb{R}^2}$ is the 2-dimensional Euclidean metric and $\overline{x}$ and $\overline{y}$ are comparison points of x, y in a comparison triangle $\overline{\triangle}$ in $\mathbb{R}^2$ of $\triangle$.

We denote by $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$ the set of all connected, path-connected, geodesic, and $\mathrm{CAT}(0)$ compact metric spaces in $\mathcal{M}$, respectively.

- $\mathcal{C}$: the set of all **connected** compact metric spaces in $\mathcal{M}$.
- $\mathcal{P}$: the set of all **path-connected** compact metric spaces in $\mathcal{M}$.
- $\mathcal{G}$: the set of all **geodesic** compact metric spaces in $\mathcal{M}$.
- $\mathcal{Z}$: the set of all $\mathrm{CAT}(0)$ compact metric spaces in $\mathcal{M}$.

Theorem 4 ([I])

Let $\mathcal{S}$ be any one of $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let H be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j. Then, there exists a topological embedding $\Phi : H \to \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$.

A **geodesic segment** is a image of a geodesic.
A metric space (X, d) is said to be a *metric tree* or $\mathbb{R}$-*tree* if it is a geodesic space and if geodesic segments G_1 and G_2 connecting x, y and y, z with $G_1 \cap G_2 = \{y\}$ satisfies that $G_1 \cup G_2$ is a geodesic segment connecting x and z for all distinct $x, y, z \in X$.
To prove our results, we first discuss the basic properties of metric trees. A metric space (X, d) is said to be 0-*hyperbolic* or satisfy the *four point condition* if

$$d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t),\ d(y, z) + d(x, t)\}$$

for all $x, y, z, t \in X$.

Proposition 1

A metric space is a metric tree if and only if it is connected and 0-*hyperbolic.*

We denote by $\mathcal{T}$ the set of all metric trees in $\mathcal{M}$, i.e., the set of all compact metric trees.

Theorem 5 ([I])

Let $n \in \mathbb{Z}_{\geq 1}$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in $\mathcal{T}$ such that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j. Let H be a compact metric space and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H. Then, there exists a topological embedding $\Phi : H \to \mathcal{T}$ such that $\Phi(v_i) = (X_i, d_i)$.

Theorem 6 ([I])

Let $\mathcal{S}$ be any one of

- $\mathcal{D}(l, a_1, a_2, a_3, a_4)$ *for some numbers* $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$,
- $\mathcal{U}$,
- $\mathcal{C}$,
- $\mathcal{P}$,
- $\mathcal{G}$,
- $\mathcal{Z}$,
- $\mathcal{T}$.

Let $n \in \mathbb{Z}_{\geq 1}$. Let H be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j. Then, there exists a topological embedding $\Phi : H \to \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$.

Part2: Proof of Theorems

Sketch of Proof of Theorem 6

1. If $\mathcal{S} = \mathcal{D}(I, a_1, a_2, a_3, a_4)$ or $\mathcal{U})$, then $\mathcal{S}$ is invariant under the direct sum.
2. If $\mathcal{S} =\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, or $\mathcal{Z}$, then $\mathcal{S}$ is invariant under the ℓ^2-product.
3. If $\mathcal{S} = \mathcal{T}$, then $\mathcal{S}$ is invariant under the wedge sum.

1. $(\coprod_{i=1}^{n+1} X_i, \coprod_{i=1}^{n+1} d_i)$
2. $(\prod_{i=1}^{n+1} X_i, \prod_{i=1}^{n+1} d_i)$
3. $(\bigvee_{i=1}^{n+1} X_i, \bigvee_{i=1}^{n+1} d_i)$

Basic idea of construction

For each $i \in \{1, \ldots, n+1\}$, we take $\zeta_i : H \to [0, 1]$ such that $\zeta_i^{-1}(0) = \{v_k \mid k \neq i\}$ and $\zeta^{-1}(1) = \{v_i\}$. We define maps $f_1, f_2, f_3 : H \to \mathcal{S}$.

1. ($\mathcal{S} = \mathcal{D}(I, a_1, a_2, a_3, a_4)$ or $\mathcal{U}$): $f_1(s) = (\coprod_{i=1}^{n+1} X_i, \coprod_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.
2. ($\mathcal{S} =$$\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, or $\mathcal{Z}$): $f_2(s) = (\prod_{i=1}^{n+1} X_i, \prod_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.
3. ($\mathcal{S} = \mathcal{T}$): $f_3(s) = (\bigvee_{i=1}^{n+1} X_i, \bigvee_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.

We see that f_1, f_2, f_3 are continuous.

If we could prove the injectivity, the proof would finish. It seems to be difficult.

Identifiers

For the injectivity, we prepare families of compact metric spaces indexed by $\mathbf{Q}$.
Let $\mathbf{Q}$ be the Hilbert cube, i.e., $\mathbf{Q} = [0, 1]^{\aleph_0}$.

Proposition 2 ([I])

There exists a family $\{(U_q, e_q)\}_{q \in \mathbf{Q}}$ of compact metric space such that

1. *$U : \mathbf{Q} \to \mathcal{M}$ defined by $U(q) = (U_q, e_q)$ is continuous;*
2. *Each (U_q, e_q) is countable and has only one accumulation point;*
3. *Let $K, L > 0$ and $q, r \in \mathbf{Q}$. If $(U_q, K \cdot e_q)$ and $(U_r, L \cdot e_r)$ are isometric to each other, then $q = r$.*

Proposition 3 ([I])

There exists a family $\{(\Upsilon_q, R_q)\}_{q\in\mathbf{Q}}$ *of metric trees such that*

1. $\Upsilon : \mathbf{Q} \to \mathcal{M}$ *defined by* $\Upsilon(q) = (\Upsilon_q, R_q)$ *is continuous;*
2. *Each* (Υ_q, R_q) *is a compact metric tree;*
3. *Let* $K, L > 0$ *and* $q, r \in \mathbf{Q}$. *If* $(\Upsilon_q, K \cdot R_q)$ *and* $(\Upsilon_r, L \cdot R_r)$ *are isometric to each other, then* $q = r$.

Modifications of f_1, f_2, f_3

For each $i \in \{1, \dots, n+1\}$, we take $\zeta_i : H \to [0,1]$ such that $\zeta_i^{-1}(0) = \{v_k \mid k \neq i\}$ and $\zeta^{-1}(1) = \{v_i\}$. We define a map $f_1, f_2, f_3 : H \to \mathcal{S}$.

1. ($\mathcal{S} = \mathcal{D}(I, a_1, a_2, a_3, a_4)$ or $\mathcal{U}$): $f_1(s) = (\coprod_{i=1}^{n+1} X_i, \coprod_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.
2. ($\mathcal{S} =$$\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, or $\mathcal{Z}$): $f_2(s) = (\prod_{i=1}^{n+1} X_i, \prod_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.
3. ($\mathcal{S} = \mathcal{T}$): $f_3(s) = (\bigvee_{i=1}^{n+1} X_i, \bigvee_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.

We see that f, g, h are continuous.

Maps F_1, F_2, F_3

Let $\xi : H \to [0,1]$ such that $\xi^{-1}(0) = \{v_i \mid i \in \{1, \dots, n+1\}\}$. Take a topological embedding $\rho : H \to \mathbf{Q}$.

1. ($\mathcal{S} = \mathcal{D}(I, a_1, a_2, a_3, a_4)$ or $\mathcal{U}$): Let (C, d_C) be a Cantor space in $\mathcal{D}(0,0,0,0,0)$. For each $s \in H$ and for each $i \in \{1 \dots n+1\}$, Put $(Y_i(s), D_i(s)) = (X_i \times C, d_i \times \xi(s) d_C)$. We define $F_1 : H \to \mathcal{S}$ by $F_1(s) = (U_{\rho(s)} \sqcup \coprod_i^{n+1} Y_i(s), \xi(s) e_{\rho(s)} \sqcup \coprod_{i=1}^{n+1} D_i(s))$.
2. ($\mathcal{S} =\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, or $\mathcal{Z}$): Take $p_i \in X_i$. We define $F_2 \to \mathcal{S}$ by $F_2(s) = (\Upsilon_{\rho(s)} \vee \prod_{i=1}^{n+1} X_i, \xi(s) R_{\rho(s)} \vee \prod_{i=1}^{n+1} \zeta_i \cdot d_i)$.
3. ($\mathcal{S} = \mathcal{T}$): $F_3(s) = (\Upsilon_{\rho(s)} \vee \bigvee_{i=1}^{n+1} X_i, \xi(s) R_{\rho(s)} \vee \bigvee_{i=1}^{n+1} \zeta_i(s) \cdot d_i)$.

Then F_1, F_2, F_3 are continuous, and for each $i \in \{1, \dots, n+1\}$, $F_1(v_i) = F_2(v_i) = F_3(v_i) = (X_i, d_i)$.

Proposition 4

the restrictions of F_1 F_2*, and* F_3 *on the set* $H \setminus \{v_i \mid i \in \{1, \dots, n+1\}\}$ *are injective, i.e., for all* $s, t \in H \setminus \{v_i \mid i \in \{1, \dots, n+1\}\}$,

1. *if* $F_1(s) = F_1(t)$ *in* $\mathcal{M}$*, i.e.,* $F(s)$ *and* $F(t)$ *are isometric to each other, then* $s = t$.
2. *if* $F_2(s) = F_2(t)$ *in* $\mathcal{M}$*, i.e.,* $G(s)$ *and* $G(t)$ *are isometric to each other, then* $s = t$.
3. *if* $F_3(s) = F_3(t)$ *in* $\mathcal{M}$*, i.e.,* $H(s)$ *and* $H(t)$ *are isometric to each other, then* $s = t$.

Remark

Actually, we need to more modifications for H.

Proof of Proposition for F_1

$F_1(s) = (U_{\rho(s)} \sqcup \coprod_{i}^{n+1} Y_i(s), \xi(s) e_{\rho(s)} \sqcup \coprod_{i=1}^{n+1} D_i(s))$.
$F_1(t) = (U_{\rho(t)} \sqcup \coprod_{i}^{n+1} Y_i(t), \xi(t) e_{\rho(t)} \sqcup \coprod_{i=1}^{n+1} D_i(t))$.
$(Y_i(s), D_i(s)) = (X_i \times C, d_i \times \xi(s) d_C)$.
Since C is perfect, and U_q is homeomorphic to the one-point compactification of the countable discrete space, we have $(U_{\rho(s)}, \xi(s) e_{\rho(s)})$ and $(U_{\rho(t)}, \xi(t) e_{\rho(t)})$ are isometric to each other. Thus, $\rho(s) = \rho(t)$, and hence $s = t$.

Proof of Proposition for F_2

$F_2(s) = (\Upsilon_{\rho(s)} \vee \prod_{i=1}^{n+1} X_i, \xi(s)R_{\rho(s)} \vee \prod_{i=1}^{n+1} \zeta_i \cdot d_i)$.
$F_2(t) = (\Upsilon_{\rho(t)} \vee \prod_{i=1}^{n+1} X_i, \xi(t)R_{\rho(t)} \vee \prod_{i=1}^{n+1} \zeta_i \cdot d_i)$.

Lemma 1

Let X and Y be perfect Hausdorff spaces. If $S \subset X \times Y$ is homeomorphic to $\mathbb{R}$, then S has no interior points.

By the lemma, we see that $(\Upsilon, \xi(s) \cdot R_{\rho(s)})$ $(\Upsilon, \xi(t) \cdot R_{\rho(t)})$ are isometric to each other. Thus, $\rho(s) = \rho(t)$, and hence $s = t$.

Topological embeddings into GH sp.

We want to show the injectivety of F_1, F_2, F_3, but it seems to be difficult.
In the argument explained above, we replace H with $\coprod_{k=1}^{N} H \times \{k\}$, where $N > n+1$. Then, We have a continuous map $F_1, F_2, F_3 : \coprod_{k=1}^{N} H \times \{k\} \to \mathcal{M}$ such that

1. For each $i \in \{1, \dots, n+1\}$, and for $k \in \{1, \dots, N\}$, we have $F_1((v_i, k)) = F_2((v_i, k)) = F_3((v_i, k)) = (X_i, d_i)$.
2. Let $s, t \in H \setminus \{v_i \mid i \in \{1, \dots, n+1\}\}$, and $k, l \in \{1, \dots, N\}$.
   If $F_1((s, k)) = F_1((t, l))$, then $(s, k) = (t, l)$.
   If $F_2((s, k)) = F_2((t, l))$, then $(s, k) = (t, l)$.
   If $F_3((s, k)) = F_3((t, l))$, then $(s, k) = (t, l)$.

We now prove that there exists $k \in \{1, \ldots, N\}$ such that $F_1 : H \times \{k\} \to \mathcal{S}$ is injective.
For the sake of contradiction, we suppose that for all k, the map $F_1 : H \times \{k\} \to \mathcal{S}$ is not injective. Then for each k, there exists $i \in \{1, \ldots, n+1\}$ and $s \in H$ such that $F_1(s, k) = (X_i, d_i)$.

Since $N > n+1$, by the pigeonhole principle, there exist $i \in \{1, \dots, n+1\}$, $k, l \in \{1, \dots, N\}$, and $s, t \in H$ we have $F_1(s, k) = (X_i, d_i)$ and $F_1(t, l) = (X_i, d_i)$. This is a contradiction.

Therefore there exists $k \in \{1, \dots, N\}$ such that $F_1 : H \times \{k\} \to \mathcal{S}$ is injective. Similarly, we obtain a injective map $F_2 : H \times \{k'\} \to \mathcal{S}$ and $F_3 : H \times \{k''\} \to \mathcal{S}$ for some $k', k'' \in \{1, \dots, N\}$. They are maps as required.
This finishes the proof.

There are analogues of embeddings into $\mathcal{S} = \mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, $\mathcal{Z}$, and $\mathcal{T}$ for the **pointed Gromov–Hausdorff space**.

Thank you very much for your attention.