

Continua in the Gromov–Hausdorff space

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In this talk, for every compact metrizable space X , I construct topological embeddings from X into

- ① the set of all connected spaces;
- ② the set of all path-connected spaces;
- ③ the set of all CAT(0) spaces

in the **Gromov–Hausdorff space**.

This talk is based on my paper

- Yoshito Ishiki, *Continua in the Gromov–Hausdorff space*, Topology Appl. 312 (2022), 10858.

The Gromov–Hausdorff space

The Gromov–Hausdorff space is a moduli space of all compact metric spaces, and it can be considered as a “universal” generalization of the Hausdorff hyperspace equipped with the Hausdorff distance.

Definition 1

For a metric space (Z, h) , and for subsets A, B of (Z, h) , we denote by $\mathcal{H}(A, B; Z, h)$ the Hausdorff distance of A and B in Z . For metric spaces (X, d) and (Y, e) , the **Gromov–Hausdorff distance** $\mathcal{GH}((X, d), (Y, e))$ between (X, d) and (Y, e) is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z, h)$, where (Z, h) is a metric space, and $i : (X, d) \rightarrow (Z, h)$ and $j : (Y, e) \rightarrow (Z, h)$ are isometric embeddings.

We denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the **Gromov–Hausdorff space**.

The Gromov–Hausdorff space

- $\mathcal{GH}$ is actually a metric on $\mathcal{M}$.

Theorem A (Ivanov–Nikolaeva–Tuzhilin (2016),
Chowdhury–Mémoli (2018))

The space $(\mathcal{M}, \mathcal{GH})$ is a geodesic space, i.e., for all (X_0, d_0) and (X_1, d_1) , there exists compact metric spaces $\{(X_t, d_t)\}_{t \in [0,1]}$ such that $\mathcal{GH}((X_t, d_t), (X_s, d_s)) = |t - s| \cdot \mathcal{GH}((X_0, d_0), (X_1, d_1))$.

Theorem B ([I])

For all distinct $(X, d), (Y, e) \in \mathcal{M}$, there exist $2^{\aleph_0}$ many geodesics between (X, d) and (Y, e) .

Motivation–Embeddings

Question

Can all separable metric spaces be isometrically embedded into the Gromov–Hausdorff space?

- The case of finite metric spaces were positively solved. (Ivanov–Tuzhilin, 2019).
- Even the case of compact metric spaces are not known yet.
- Other cases are not known.
- Ultrametric analogues of this question were positively perfectly solved. (Wan, 2021).
- It is known that the identity map of $\mathcal{M}$ is an only isometric automorphism on $(\mathcal{M}, \mathcal{GH})$. In particular, it is not isometric to the Urysohn universal space (Ivanov–Tuzhilin, 2019).
- As far as the speaker knows, it is not known the homeomorphism type of $\mathcal{M}$.

As a consequence of the speaker's theorem on the existence of branching geodesics in the Gromov–Hausdorff space, the speaker proved:

Let $\mathbf{Q} = [0, 1]^{\aleph_0}$ (the Hilbert cube).

Theorem C ([I])

Let $\mathcal{S}$ be any one of the following subsets of $\mathcal{M}$

- ① $\mathcal{M}$
- ② the set of all compact metric spaces with infinite topological dimension.
- ③ the set of all compact metric spaces homeomorphic to the Cantor set.

Then for all $(X, d), (Y, e)$ in $\mathcal{S}$, there exists a topological embedding $I : \mathbf{Q} \rightarrow \mathcal{S}$ such that the image of I contains (X, d) and (Y, e) . In particular, we see that the set $\mathcal{S}$ is infinite-dimensional.

The speaker want to know the size (dimension) of subsets of $\mathcal{M}$.

- $\mathcal{D}(l, a_1, a_2, a_3, a_4)$: the set of all compact metric spaces in $\mathcal{M}$ satisfying

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

- $\mathcal{U}$: the set of all compact **ultrametric** spaces in $\mathcal{M}$.
- $[0, 1]^{\aleph_0}$: A Hilbert cube.

Theorem D ([I])

Assume that $\mathcal{S} = \mathcal{D}(l, a_1, a_2, a_3, a_4)$ for some numbers $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$ or $\mathcal{S} = \mathcal{U}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let H be a Hilbert cube, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H . Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be compact metric spaces in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j . Then there exists a topological embedding $\Phi : H \rightarrow \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$ for all $i \in \{1, \dots, n+1\}$.

Main results

We denote by $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$ the set of all connected, path-connected, geodesic, and CAT(0) compact metric spaces in $\mathcal{M}$, respectively.

Theorem 1 ([I])

Let $\mathcal{S}$ be any one of $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$. Let $n \in \mathbb{Z}_{\geq 1}$ and H be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in H . Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct i, j . Then, there exists a topological embedding $\Phi : H \rightarrow \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$.

Corollary 1 ([I])

The sets $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$ are path-connected and their all non-empty open subsets have infinite topological dimension.

Sketch of proof

We consider the case of $\mathcal{S} = \mathcal{C}$.

H : an arbitrary compact ultrametric.

$v_1, \dots, v_{n+1} \in H$: mutually distinct.

$(X_1, d_1), \dots, (X_{n+1}, d_{n+1})$ mutually distinct points in $\mathcal{C}$.

For each $i \in \{1, \dots, n+1\}$, take a continuous function

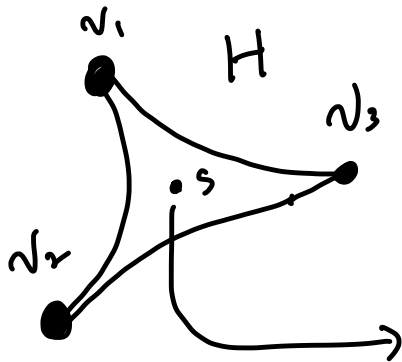
$\zeta_i : H \rightarrow [0, 1]$ such that $\zeta_i^{-1}(0) = \{v_j \mid j \neq i\}$ and $\zeta_i^{-1}(1) = \{v_i\}$.

We also take a continuous function $\xi : H \rightarrow [0, 1]$ with

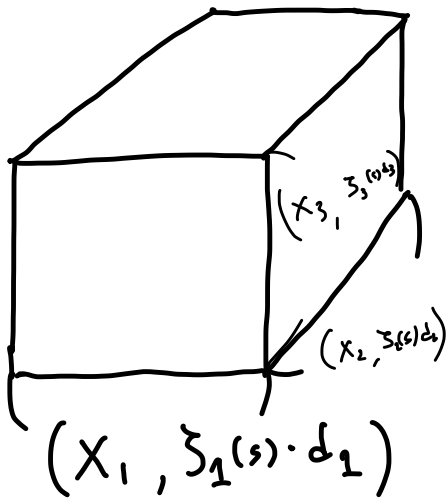
$\xi^{-1}(0) = \{v_i \mid i = 1, \dots, n+1\}$.

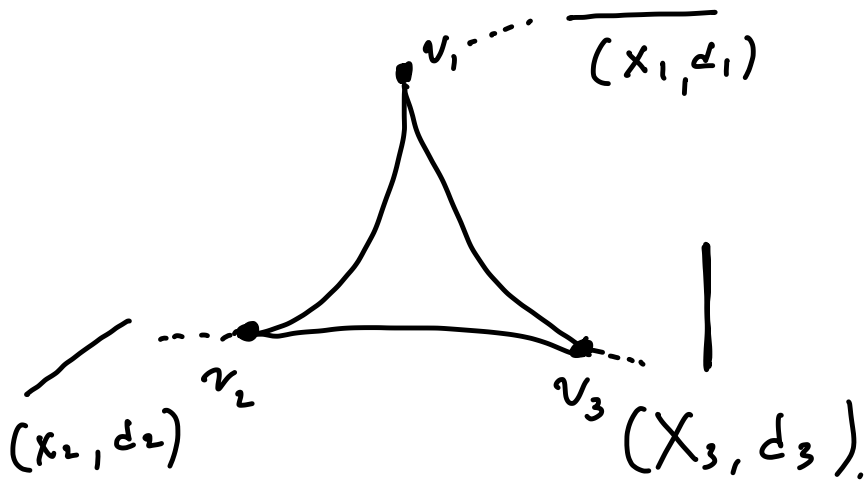
We define $P = \prod_{i=1}^{n+1} X_i$. For each $s \in H$, we define a pseudo-metric E_s on P by

$$E_s(x, y) = \sqrt{\sum_{i=1}^{n+1} (\zeta_i(s) \cdot d_i(x_i, y_i))^2}.$$



$$\left(\prod_{i=1}^3 X_i, \sqrt{\sum_{i=1}^3 \zeta_i(s) d_i} \right)$$





Identifier

Since H is compact metrizable, we can regard H as a closed subset of the Hilbert cube $\mathbf{Q}$, which is homeomorphic to $\prod_{i=1}^{\infty} [2^{-2i}, 2^{-2i+1}]$. We can define a family of compact metric trees $\{(\Upsilon, R[\mathbf{a}])\}_{\mathbf{a} \in \mathbf{Q}}$ such that

- 1 If $\mathbf{a}, \mathbf{b} \in \mathbf{Q}$ and $K, L \in (0, \infty)$ satisfy that $(\Upsilon, K \cdot R[\mathbf{a}])$ is isometric to $(\Upsilon, L \cdot R[\mathbf{b}])$, then $\mathbf{a} = \mathbf{b}$.
- 2 the map from $\mathbf{Q}$ into $\mathcal{M}$ defined by $\mathbf{a} \mapsto (\Upsilon, R[\mathbf{a}])$ is continuous.

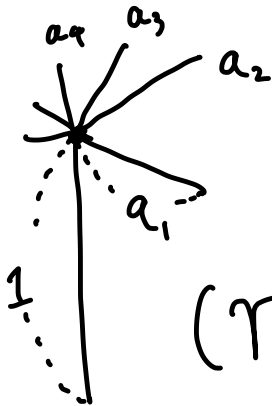
Remark

We can consider that $H \subset \mathbf{Q}$.

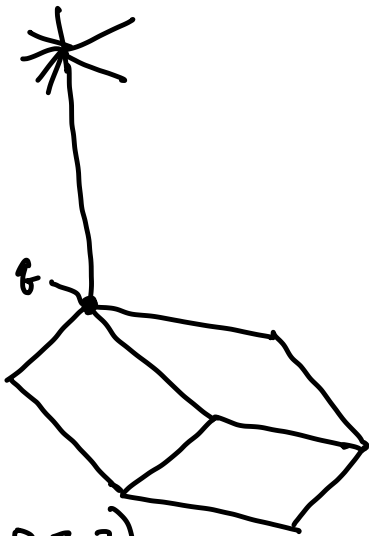
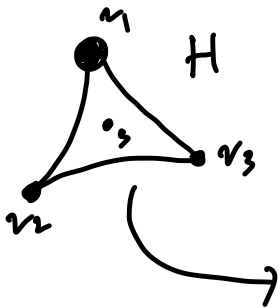
$$\mathcal{A} = \{a_i\}_{i \in \mathbb{N}_{20}}. \quad 1 \succ a_0 \succ a_1 \succ \dots$$

$a_i \rightarrow 0, i \rightarrow \infty.$

$\mathcal{A} \nearrow \mathcal{A}$



$$(\tau, R[\mathcal{A}]).$$



$$(P_{\mathcal{V}}^{\mathcal{T}}, E_s V_{\mathcal{V}}(\mathbb{Z}(s) \cdot R[s])).$$

Lemma 1

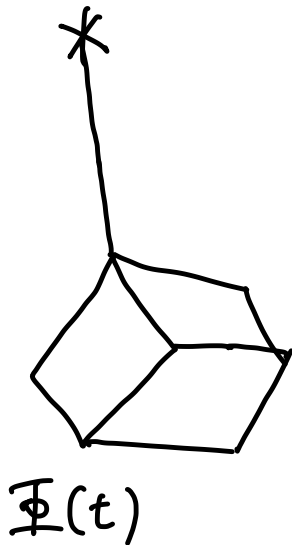
Let X and Y be Hausdorff topological spaces possessing no isolated points. Then any topological arc in the product space $X \times Y$ has no interior points.

We now construct a topological embedding from H into $\mathcal{C}$.

$s, t \in H \setminus \{v_1, v_2, v_3\}$.



$\cong$
isom



Thank you very much for your attention.