# Metric trees in the Gromov–Hausdorff space

**Yoshito Ishiki** (伊敷喜斗)

**RIKEN, Photonics Control Technology Team**

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In this talk, for every compact metrizable space $X$, I construct topological embeddings from $X$ into the set of all metric trees in the Gromov–Hausdorff space.

This talk is based on my paper

- Yoshito Ishiki, *Metric trees in the Gromov–Hausdorff space*, to appear in Comment. Math. Univ. Carol.

# The Gromov–Hausdorff space

The Gromov–Hausdorff space is a moduli space of all compact metric spaces, and it can be considered as a “universal” generalization of the Hausdorff hyperspace equipped with the Hausdorff distance.

## Definition 1

For a metric space $(Z, h)$, and for subsets $A$, $B$ of $(Z, h)$, we denote by $\mathcal{H}(A, B; Z, h)$ the Hausdorff distance of $A$ and $B$ in $Z$. For metric spaces $(X, d)$ and $(Y, e)$, the **Gromov–Hausdorff distance** $\mathcal{GH}((X, d), (Y, e))$ between $(X, d)$ and $(Y, e)$ is defined as the infimum of all values $\mathcal{H}(i(X), j(Y); Z, h)$, where $(Z, h)$ is a metric space, and $i : (X, d) \to (Z, h)$ and $j : (Y, e) \to (Z, h)$ are isometric embeddings.

We denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the **Gromov–Hausdorff space**.

# The Gromov–Hausdorff space

- $\mathcal{GH}$ is actually a metric on $\mathcal{M}$.

**Theorem A (Ivanov–Nikolaeva–Tuzhilin (2016), Chowdhury–Mémoli (2018) )**

The space $(\mathcal{M}, \mathcal{GH})$ is a geodesic space, i.e., for all $(X_0, d_0)$ and $(X_1, d_1)$, there exists compact metric spaces $\{(X_t, d_t)\}_{t\in[0,1]}$ such that $\mathcal{GH}((X_t, d_t), (X_s, d_s)) = |t - s| \cdot \mathcal{GH}((X_0, d_0), (X_1, d_1))$.

**Theorem B ([I])**

For all distinct $(X, d), (Y, e) \in \mathcal{M}$, there exist $2^{\aleph_0}$ many geodesics between $(X, d)$ and $(Y, e)$.

# Motivation–Embeddings

**Question**

Can all separable metric spaces be isometrically embedded into the Gromov–Hausdorff space?

- The case of finite metic spaces were positively solved. (Ivanov–Tuzhilin, 2019).
- Even the case of compact metric spaces are not known yet.
- Other cases are not known.
- Ultrametric analogues of this question were positively perfectly solved. (Wan, 2021).
- It is know that the identity map of $\mathcal{M}$ is an only isometric automorphism on $(\mathcal{M}, \mathcal{GH})$. In particular, it is not isometric to the Urysohn universal space (Ivanov–Tuzhilin, 2019).
- As far as the speaker knows, it is not known the homeomorphism type of $\mathcal{M}$.

As a consequence of the speaker's theorem on the existence of branching geodesics in the Gromov–Hausdorff space, the speaker proved:
Let $\mathbf{Q} = [0,1]^{\aleph_0}$ ( the Hilbert cube).

## Theorem C ([I])

Let $\mathcal{S}$ be any one of the following subsets of $\mathcal{M}$

1. $\mathcal{M}$
2. the set of all compact metric spaces with infinite topological dimension.
3. the set of all compact metric spaces homeomorphic to the Cantor set.

Then for all $(X, d)$, $(Y, e)$ in $\mathcal{S}$, there exists a topological embedding $I : \mathbf{Q} \to \mathcal{S}$ such that the image of $I$ contains $(X, d)$ and $(Y, e)$. In particular, we see that the set $\mathcal{S}$ is infinite-dimensional.

The speaker want to know the size (dimension) of subsets of $\mathcal{M}$.

- $\mathcal{D}(l, a_1, a_2, a_3, a_4)$: the set of all compact metric spaces in $\mathcal{M}$ satisfying

$$\dim_{\mathrm{T}} X = l, \dim_{\mathrm{H}}(X, d) = a_1, \dim_{\mathrm{P}}(X, d) = a_2, \overline{\dim}_{\mathrm{B}}(X, d) = a_3, \dim_{\mathrm{A}}(X, d) = a_4.$$

- $\mathcal{U}$: the set of all compact **ultrametric** spaces in $\mathcal{M}$.
- $[0,1]^{\aleph_0}$: A Hilbert cube.

## Theorem D ([I])

Assume that $\mathcal{S} = \mathcal{D}(l, a_1, a_2, a_3, a_4)$ for some numbers $(l, a_1, a_2, a_3, a_4) \in \mathcal{R}$ or $\mathcal{S} = \mathcal{U}$. Let $n \in \mathbb{Z}_{\geq 1}$. Let $H$ be a Hilbert cube, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in $H$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be compact metric spaces in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct $i, j$. Then there exists a topological embedding $\Phi : H \to \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$ for all $i \in \{1, \ldots, n+1\}$.

We denote by $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$ the set of all connected, path-connected, geodesic, and $\mathrm{CAT}(0)$ compact metric spaces in $\mathcal{M}$, respectively.

## Theorem E ([I])

Let $\mathcal{S}$ be any one of $\mathcal{C}$, $\mathcal{P}$, $\mathcal{G}$, and $\mathcal{Z}$. Let $n \in \mathbb{Z}_{\geq 1}$ and $H$ be a compact metrizable space, and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in $H$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in $\mathcal{S}$ satisfying that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct $i, j$. Then, there exists a topological embedding $\Phi : H \to \mathcal{S}$ such that $\Phi(v_i) = (X_i, d_i)$.

## Theorem F ([I])

The subsets of the GH space appearing in the previous theorems are path-connected and their all non-empty open subsets have infinite topological dimension.

# Main results

We denote by $\mathcal{T}$ the set of all metric trees in $\mathcal{M}$.

## Theorem 1

*Let $n \in \mathbb{Z}_{\geq 1}$. Let $\{(X_i, d_i)\}_{i=1}^{n+1}$ be a sequence in $\mathcal{T}$ such that $\mathcal{GH}((X_i, d_i), (X_j, d_j)) > 0$ for all distinct $i, j$. Let $H$ be a compact metric space and $\{v_i\}_{i=1}^{n+1}$ be $n+1$ different points in $H$. Then, there exists a topological embedding $\Phi : H \to \mathcal{T}$ such that $\Phi(v_i) = (X_i, d_i)$.*

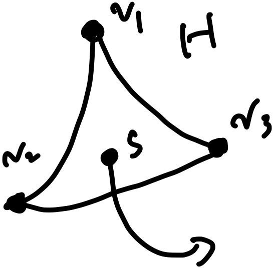
## Corollary 1

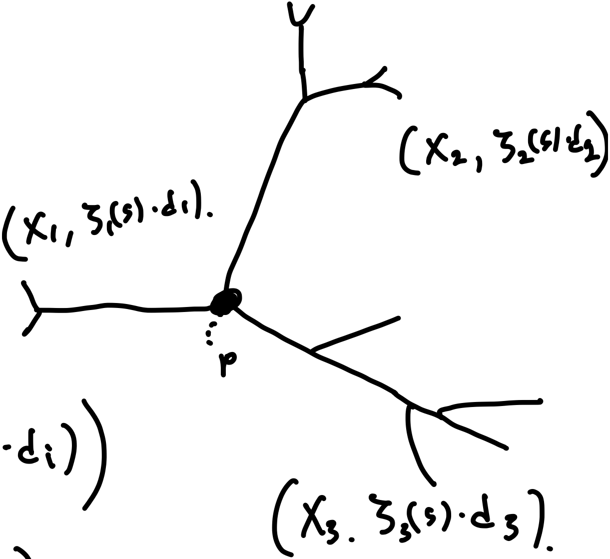
*The set $\mathcal{T}$ is path-connected and its all non-empty open subsets have infinite topological dimension.*

# Sketch of proof

$H$: an arbitrary compact ultrametric.
$v_1, \dots, v_{n+1} \in H$: mutually distinct.
$(X_1, d_1), \dots, (X_{n+1}, d_{n+1})$ mutually distinct points in $\mathcal{C}$.
For each $i \in \{1, \dots, n+1\}$, take a continuous function $\zeta_i : H \to [0,1]$ such that $\zeta_i^{-1}(0) = \{\, v_j \mid j \neq i \,\}$ and $\zeta_i^{-1}(1) = \{v_i\}$.
We also take a continuous function $\xi : H \to [0,1]$ with $\xi^{-1}(0) = \{\, v_i \mid i = 1, \dots, n+1 \,\}$.

We use wedge sum of metric spaces.

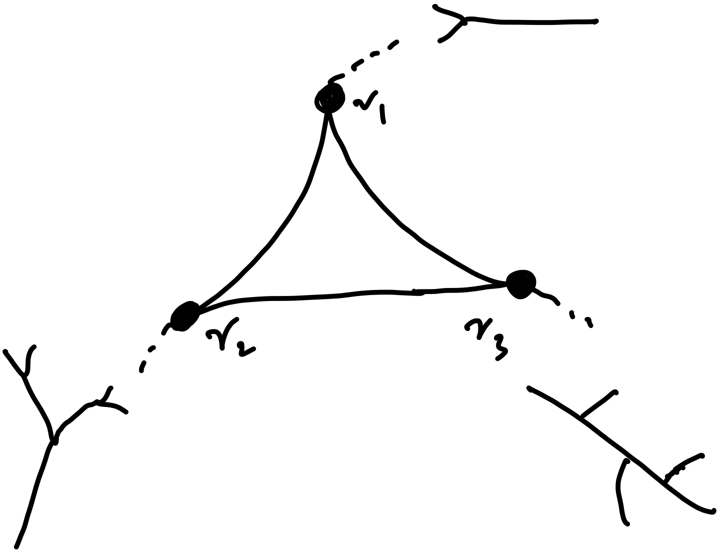



$$\left(\bigvee_p X_i, \bigvee_p (\zeta_i \cdot d_i)\right)$$

$$\parallel$$

$$(P, E_s)$$


$v_1$
$v_2$
$v_3$

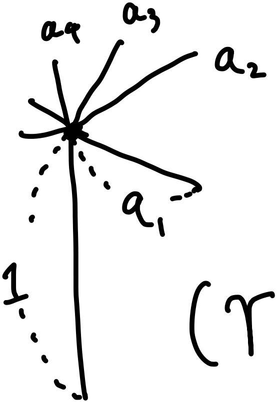
# Identifyer

Since $H$ is compact metrizable, we can regard $H$ as a closed subset of the Hilbert cube $\mathbf{Q}$, which is homeomorphic to $\prod_{i=1}[2^{-2i}, 2^{-2i+1}]$. We can define a family of compact metric trees $\{(\Upsilon, R[\mathbf{a}])\}_{\mathbf{a}\in\mathbf{Q}}$ such that

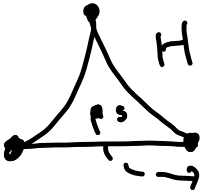
1. If $\mathbf{a}, \mathbf{b} \in \mathbf{Q}$ and $K, L \in (0, \infty)$ satisfy that $(\Upsilon, K \cdot R[\mathbf{a}])$ is isometric to $(\Upsilon, L \cdot R[\mathbf{b}])$, then $\mathbf{a} = \mathbf{b}$.
2. the map from $\mathbf{Q}$ into $\mathcal{M}$ defined by $\mathbf{a} \mapsto (\Upsilon, R[\mathbf{a}])$ is continuous.

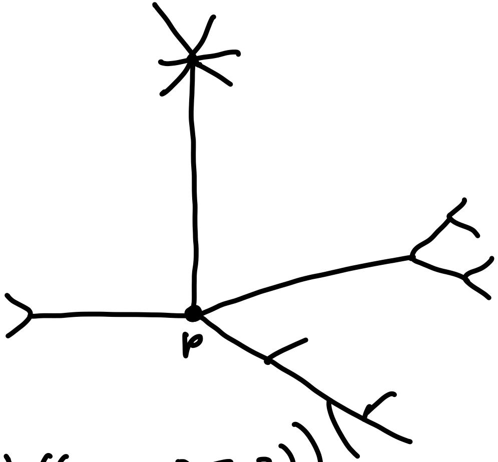
$\mathfrak{a} = \{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$. $1 > a_0 > a_1 > \cdots$.

$a_i \to 0, \ i \to \infty$.

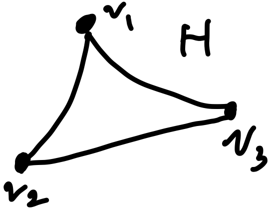


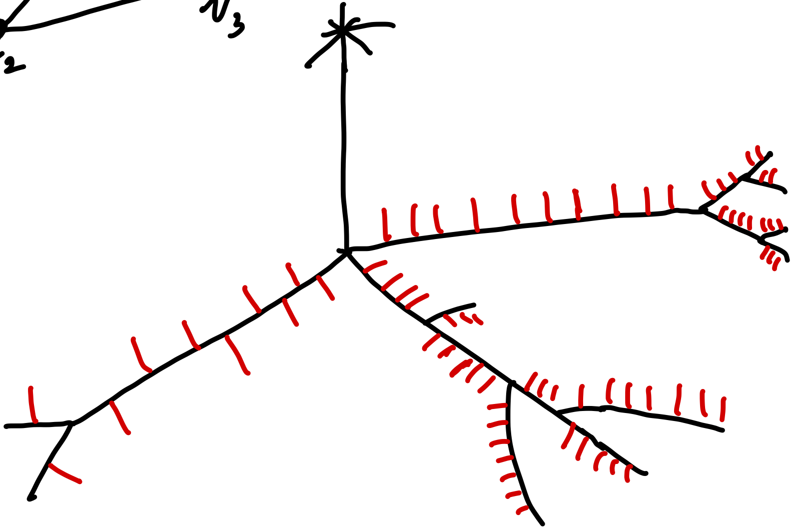

$(\Upsilon, R[\mathfrak{a}])$.

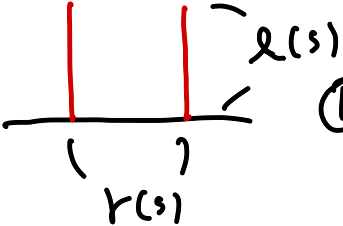



$$\left(P \vee_{p} \Upsilon,\ E_s \bigvee_{p} \left(\xi(s) \cdot R[s]\right)\right).$$


$v_1$
H
$v_3$
$v_2$





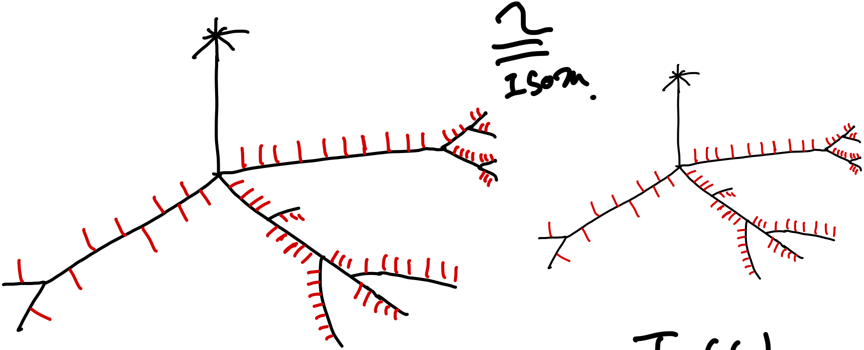

① $\ell(s) \to 0$ as $s \to v_i$.

② $\ell(s) < \frac{1}{16}\xi(s)$

$r(s) < \frac{1}{16}\xi(s)$

for all $s \in H \setminus \{v_1, v_2, v_3\}$.

$s, t \in H \setminus \{v_1, v_2, v_3\}.$




$\Phi(s)$ $\Phi(t)$

Thank you very much for your attention.