Non-Archimedean Urysohn universal metric spaces

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Part 0: Introduction, and my research

Definition 1 (Metric space)

For a set X, a map $d\colon X \times X \to [0, \infty)$ is **metric** if

(1) $d(x, y) = d(y, x)$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$.

A topological space X is **metrizable** if there exists a metric d that generates the same topology of X.

- I study **Geometry of metric spaces.**
- Geometry......properties determined by metrics

- My research is **Geometry of giant metric spaces** (moduli spaces, and universal spaces).
- My research can be considered as one of the attempts to answer Ultimate Question.

Ultimate Question

Clarify all (geometric) properties of all metric spaces.

It is impossible to solve this question, however we can get close to an answer of it.

Giant metric spaces

(1) **Moduli spaces** (Space of geometric structures on metrizable spaces)
We can consider that spaces of geometric structures contain all information of of metric spaces.
  1.1 Space of metrics $\mathrm{Met}(X)$ (Moduli spaces of all metrics on a fixed metrizable space).
  1.2 Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$ (Moduli space of all compact metric spaces).

(2) **Universal spaces: Our Subject!**
A metric space contains all separable metric spaces can be considered to reflect all properties of all separable metric spaces.
  2.1 Urysohn universal (ultrametric) space $(\mathbb{V}_R, \sigma_R)$.

Ultrametric analogues

Definition 2 (Ultrametric spaces)

A metric d on a set X is an **ultrametric** or **non-Archimedean metric** if

$$d(x,y) \le d(x,z) \vee d(z,y)$$
$$(= \max\{d(x,z), d(z,y)\}),$$

where $\vee$ is the maximal operator on $\mathbb{R}$.

A topological space X is **ultrametrizable** if there exists an ultrametric d that generates the same topology of X.

- A metrizable space is ultrametrizable if and only if it has topological dimension 0.
- Some people (including the speaker) investigate ultrametric analogues of statements on metric spaces.

Universal spaces

There are many spaces in which any arbitrary separable metric space can be isometrically embedded. Such spaces are said to be **universal for all separable metric spaces**.

(1) The space $C([0,1],\mathbb{R})$ of all continuous functions on $[0,1]$ equipped with the supremum metric (Banach-Mazur Theorem).

(2) The Banach space ℓ^∞ of bounded real sequences (Fréchet embedding Theorem).

(3) The Urysohn universal metric space $(\mathbb{U},\rho)$.

- The Urysohn universal metric space is a universal metric space that has a strong homogeneity (ultrahomogeneity).
- I study a non-Archimedean analogue $(\mathbb{V}_R,\sigma_R)$ of the Urysohn universal ultrametric space, where R is a subset of $[0,\infty)$ with $0\in R$.

Part 1: Definitions and Constructions of the ordinary Urysohn universal space, and non-Archimedean analogues.

Injectivity of metric spaces

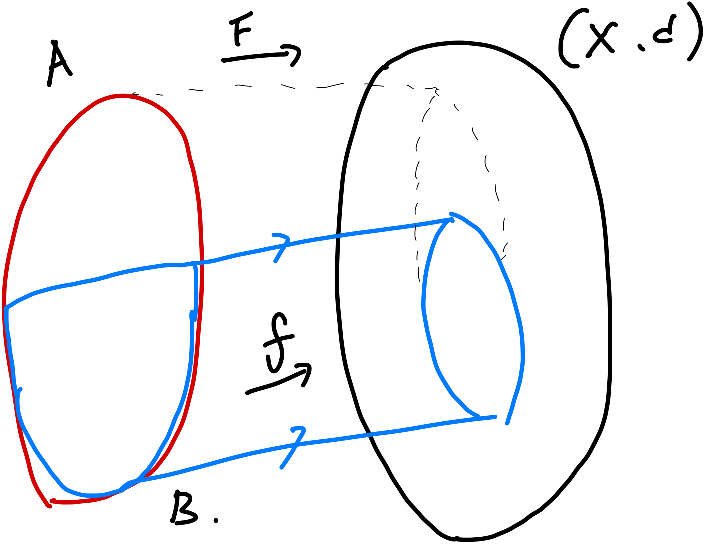
Definition 3

Let $\mathcal{C}$ be a class of metric spaces. A metric space (X, d) is $\mathcal{C}$-**injective** or **injective for** $\mathcal{C}$ if for every $(A, e) \in \mathcal{C}$, every $(B, h) \in \mathcal{C}$, and for every pair of isometries $\iota\colon (B, h) \to (A, e)$ and $f\colon (B, h) \to (X, d)$, there exists an isometric embedding $F\colon (A, e) \to (X, d)$ such that $F \circ \iota = f$.

$$\begin{tikzcd}
(A, e) \arrow[dr, dotted, "\exists F"] & \\
(B, h) \arrow[u, hook, "\iota"] \arrow[r, "f"'] & (X, d)
\end{tikzcd}$$

This is a similar concept as injective modules in homological algebra.

Figure of injectivity



We often consider the injectivity for a class of finite metric spaces.

Definition 4

We denote by $\mathcal{F}$ the class of all finite metric spaces.

Theorem A (Urysohn, 1927)

There exists a separable complete $\mathcal{F}$-injective metric space, and it is unique up to isometry.

Uniqueness follows from the so-called back-and-forth argument.

Definition 5

We denote by $(\mathbb{U}, \rho)$ a unique separable complete $\mathcal{F}$-injective metric space. We call it the **Urysohn universal metric space**.

- Our main subject is a non-Archimedean analogue of $(\mathbb{U}, \rho)$.
- $(\mathbb{U}, \rho)$ is universal for all sep. met. sps.

Let us recall the ordinary homogeneity.

Definition 6

A metric space (X, d) is said to be **homogeneous** if for every pair of points a, b in X, there exists an isometric bijection $F\colon X \to X$ such that $F(a) = b$.

The Urysohn universal space satisfies a stronger homogeneity.

Proposition 1

Let A, B be finite metric subspaces of $(\mathbb{U}, \rho)$, and $f\colon A \to B$ be an isometric bijection. Then there exists an isometric bijection $F\colon \mathbb{U} \to \mathbb{U}$ such that $F|_A = f$.

The property in Proposition 1 is called the **ultrahomogeneity**. This concept is note related to the non-Archimedean analogue of the ordinary homogeneity.

Non-Archimedean Urysohn universal spaces

Ultrametrics: $d(x,y) \le d(x,z) \vee d(z,y)$.

Definition 7

A set R is said to be a **range set** if $R \subset [0,\infty)$ and $0 \in R$.

Example 1

Ex.

(1) $\{0\}$,

(2) $[0,\infty)$,

(3) $\{0\} \cup \{2^{-n} \mid n \in \mathbb{Z}\}$,

(4) $\mathbb{Z}_{\ge 0}$,

(5) $\mathbb{Q}_{\ge 0}$,

(6) $\{0\} \cup (\mathbb{R} \setminus \mathbb{Q})$,

(7) the (middle-third) Cantor set.

Definition 8

For a range set R, an ultrametric space (X, d) is R-**valued** if $d(x, y) \in R$ for all $x, y \in X$.

Definition 9

For a range set R, we denote by $\mathcal{N}(R)$ the class of all finite R-valued ultrametric spaces.

When considering an non-Archimedean analogues of metric spaces, people often uses R-valued ultrametrics rather than ultrametrics.

$$\text{Archimedes} \underset{Analogy}{\longleftrightarrow} (R\text{-valued) Non-Archimedes}$$

$$\text{“}+\text{”} \underset{Analogy}{\longleftrightarrow} \text{“}\vee\text{”} : \text{ maximum operator}$$

$$\text{“}d(x,y) \leq d(x,z) + d(z,y)\text{”} \underset{Analogy}{\longleftrightarrow} \text{“}d(x,y) \leq d(x,z) \vee d(z,y)\text{”}$$

$$\text{Metrics} \underset{Analogy}{\longleftrightarrow} \text{Ultrametrics}$$

$$\text{finite met. sps: } \mathcal{F} \underset{Analogy}{\longleftrightarrow} \mathcal{N}(R) : R\text{-valued finite ultramet.sps}$$

$$\mathcal{F}\text{-injecitivity} \underset{Analogy}{\longleftrightarrow} \mathcal{N}(R)\text{-injectivity}$$

$$(\mathbb{U}, \rho) \underset{Analogy}{\longleftrightarrow} (\mathbb{V}_R, \sigma_R)$$

$$\text{Arens–Eells thm} \underset{Analogy}{\longleftrightarrow} \text{N. -A. Arens–Eells thm}$$

$$\text{Hausdorff's met. ext. thm} \underset{Analogy}{\longleftrightarrow} \text{Ultramet. ext. thm}$$

$$\text{Michael's conti. selec. thm} \underset{Analogy}{\longleftrightarrow} \text{0-dim Michael's conti. selec. thm}$$

Theorem B (folklore)

If a range set R is **finite or countable**, *then there exists a separable complete R-valued $\mathcal{N}(R)$-injective ultrametric space, and it is unique up to isometry.*

Remark

If R is **uncountable**, then all $\mathcal{N}(R)$-injective ultrametric spaces are not separable.

Definition 10

For a finite or countable range set R, we denote by $(\mathbb{V}_R, \sigma_R)$ a unique separable complete R-valued $\mathcal{N}(R)$-injective ultrametric space. We call it the R-**Urysohn universal ultrametric space**.

Proposition 2

Let R be a finite or countable range set.

(1) *The space $(\mathbb{V}_R, \sigma_R)$ is universal for all separable R-valued ultrametric spaces. Namely, all separable R-valued ultrametric spaces can be isometrically embedded into $(\mathbb{V}_R, \sigma_R)$.*

(2) *The space $(\mathbb{V}_R, \sigma_R)$ is ultrahomogeneous: for every pair of finite subsets A, B of $\mathbb{V}_R$, and for every isometric bijection $f\colon A \to B$, there exists an isometric bijection $F\colon \mathbb{V}_R \to \mathbb{V}_R$.*

Theorem 1

For every **finite or countable** *range set* R*, all of the following spaces are* $(\mathbb{V}_R, \sigma_R)$*, i.e., they are separable, complete, and* $\mathcal{N}(R)$*-injective.*

(1) *(well-known): The* R*-ultrametric space* $(\mathrm{G}(R, \omega_0), \triangle)$*. (Space of maps from* R *into* $\omega_0 (= \mathbb{Z}_{\geq 0})$ *whose supports are fin. or dec. seqs. conv. to* 0*).*

(2) *(Wan, 2019): The Gromov–Hausdorff* R*-ultrametric space* $(\mathcal{U}_R, \mathcal{NA})$*. (n.-A. Analogue of GH sp)*

(3) *([I], 2023): The space* $(\mathrm{C}_0(\Gamma, R), \triangledown)$ *of all continuous functions* f *from the Cantor set* Γ *into* (R, M_R) *with* $0 \in f(\Gamma)$*. (A n.-A. analogue of the Banach-Mazur theorem,* M_R*: n.-A. analog of Euclidean met.)*

(4) *([I], 2023): The ultrametric space* $(\mathrm{Cpu}(X, R), \mathcal{UD}_X^R)$ *of all* R*-valued continuous pseudo-ultrametrics on a compact ultrametrizable space* X *with an accumulation point. (An analogue of Wan's result)*

Remark

In the **Archimedean case**, the following spaces are NOT isometric to each other.

(1) $(\mathbb{U}, \rho)$

(2) The Gromov–Hausdorff space (GH sp).

(3) $C([0,1], \mathbb{R})$ (The space appearing in the Banach–Mazur theorem)

The speaker does not know whether $(\mathrm{Met}(X), \mathcal{D}_X)$ is isometric to $(\mathbb{U}, \rho)$, where X is compact metrizable, $\mathrm{Met}(X)$ is the set of all metrics on X generating the same topology on X, and $\mathcal{D}_X$ is the supremum metric.

The space $(\mathrm{C}_0(\Gamma, R), \triangledown)$

- Γ: the Cantor set. $\{0,1\}^{\mathbb{Z}_{\geq 0}}$.
- The ultrametric M_R on R: n.-A. analog of the Euclidean metrics defined by

$$M_R(x,y) = \begin{cases} x \vee y & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

$M_R(x,y) = \inf\{\epsilon \in R \mid x \leq y \vee \epsilon \text{ and } y \leq x \vee \epsilon\}.$

- We denote by $\mathrm{C}_0(\Gamma, R)$ the set of all continuous functions $f : \Gamma \to R$ such that $0 \in f(\Gamma)$.
- We define an ultrametric $\triangledown$ on $\mathrm{C}_0(\Gamma, R)$ by

$$\triangledown(f,g) = \sup_{x \in \Gamma} M_R(f(x), g(x)).$$

- Then $(\mathrm{C}_0(\Gamma, R), \triangledown)$ is isometric to $(\mathbb{V}_R, \sigma_R)$.

The ultrametric space $(\mathrm{Cpu}(X, R), \mathcal{UD}_X^R)$

- Let X be a compact ultrametrizable space with an accumulation point. Equivalently, X is an infinite closed subset of the Cantor set Γ.
- We denote by $\mathrm{Cpu}(X, R)$ the set of all R-valued continuous pseudo-ultrametrics on X.
- We define $\mathcal{UD}_X^R$ on $\mathrm{Cpu}(X, R)$ by

$$\mathcal{UD}_X^R(d, e) = \inf\{\epsilon \in R \mid \forall x, y \in X,\ e(x, y) \leq d(x, y) \vee \epsilon \text{ and } d(x, y) \leq e(x, y) \vee \epsilon\}.$$

  The ultrametric $\mathcal{UD}_X^R$ is a non-Archimedean analogue of the supremum metric on $\mathrm{Met}(X)$: $\sup_{x,y \in X} |d(x, y) - e(x, y)|$.
- Then $(\mathrm{Cpu}(X, R), \mathcal{UD}_X^R)$ is isometric to $(\mathbb{V}_R, \sigma_R)$.

Theorem 2 ([I], 2023)

For **EVERY** *range set R, all of the following spaces are isometric to each other.*

1. *The R-ultrametric space $(\mathrm{G}(R, \omega_0), \triangle)$.*
2. *The Gromov–Hausdorff R-ultrametric space $(\mathfrak{U}_R, \mathcal{NA})$.*
3. *The space $(\mathrm{C}_0(\Gamma, R), \triangledown)$ of all continuous functions f from the Cantor set Γ into (R, M_R) with $0 \in f(\Gamma)$.*
4. *The ultrametric space $(\mathrm{Cpu}(X, R), \mathcal{UD}_X^R)$ of all R-valued continuous pseudo-ultrametrics on a compact ultrametrizable space X with an accumulation point.*

The proof uses the back-and-forth argument, and transfinite induction (Isometrical characterization of the R-petaloid ultrametric space).

Remark

Based on Theorem 2, even when R is uncountable, we call the space in Theorem 2 the R-Urysohn universal ultrametric sapce, and we represent it as $(\mathbb{V}_R, \sigma_R)$.

Proposition 3 ([I], 2023)

Let R be a range set.

(1) *The space $(\mathbb{V}_R, \sigma_R)$ is universal for all separable R-valued ultrametric spaces. Namely, all separable R-valued ultrametric spaces can be isometrically embedded into $(\mathbb{V}_R, \sigma_R)$.*

(2) *The space $(\mathbb{V}_R, \sigma_R)$ is ultrahomogeneous: for every pair of finite subsets A, B of $\mathbb{V}_R$, and for every isometric bijection $f\colon A \to B$, there exists an isometric bijection $F\colon \mathbb{V}_R \to \mathbb{V}_R$.*

Part 2: Product and hyperspace of $(\mathbb{V}_R, \sigma_R)$.

Product of $(\mathbb{V}_R, \sigma_R)$.

For metric space (X, d) and (Y, e) we define a metric $d \times_\infty e$ on $X \times Y$ by $(d \times_\infty e)((x, y), (u, v)) = d(x, u) \vee e(y, v)$.

Theorem 3 ([I], Preprint 2023)

Let R be a range set. Then $(\mathbb{V}_R \times \mathbb{V}_R, \sigma_R \times_\infty \sigma_R)$ is isometric to $(\mathbb{V}_R, \sigma_R)$.

For metric space (X, d) and (Y, e), for $p \in [1, \infty)$ we define a metric $d \times_p e$ on $X \times Y$ by
$(d \times_p e)((x, y), (u, v)) = (d(x, u)^p + e(y, v)^p)^{1/p}$.

Theorem 4 ([I], Preprint 2023)

For every $p \in [1, \infty]$, the space $(\mathbb{U} \times \mathbb{U}, \rho \times_p \rho)$ is NOT isometric to $(\mathbb{U}, \rho)$.

Hyperspace of $(\mathbb{V}_R, \sigma_R)$.

For a metric space (X, d), we denote by $\mathcal{K}(X)$ the set of all non-empty compact subsets of X, and denote by $\mathcal{HD}_d$ the Hausdorff distance on $\mathcal{K}(X)$ induced by d.

Theorem 5 ([I], Preprint 2023)

Let R be a range set. Then the space $(\mathcal{K}(\mathbb{V}_R), \mathcal{HD}_{\sigma_R})$ is isometric to $(\mathbb{V}_R, \sigma_R)$.

Theorem C (Antonyan, 2020)

The Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$ is isometric to $\mathcal{K}(\mathbb{U})/\mathrm{Isom}(\mathbb{U}, \rho)$, where $\mathcal{K}(\mathbb{U})$ is the hyperspace of $\mathbb{U}$ of all non-empty compact subsets ($\mathcal{K}(\mathbb{U})/\mathrm{Isom}(\mathbb{U}, \rho)$ is the quotient space $\mathcal{K}(\mathbb{U})$ by $\mathrm{Isom}(\mathbb{U}, \rho)$).

Symmetric products of $(\mathbb{V}_R, \sigma_R)$.

For $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$ and $l \in (0, \infty]$, we also denote by $\mathcal{K}_{m,l}(X)$ the space of all $E \in \mathcal{K}(X)$ such that $\mathrm{card}(E) \leq m$ and $\mathrm{diam}(E) \leq l$, where in the case of $m = l = \infty$, the inequalities $\mathrm{card}(E) \leq \infty$ and $\mathrm{diam}(E) \leq \infty$ mean that there are no restrictions on $\mathrm{card}(E)$ and $\mathrm{diam}(E)$.
In particular, $\mathcal{K}_{\infty,\infty}(X) = \mathcal{K}(X)$.
$\mathcal{K}_{m,\infty}(X)$: the m-th symmetric product.

Theorem 6 ([I], Preprint 2023)

Let $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$, and R be a range set, $l \in (0, \infty]$. Then $(\mathcal{K}_{m,l}(\mathbb{V}_R), \mathcal{HD}_{\sigma_R})$ is the R-Urysohn universal ultrametric space. In particular, the space $(\mathcal{K}_{m,l}(\mathbb{V}_R), \mathcal{HD}_{\sigma_R})$ is isometric to $(\mathbb{V}_R, \sigma_R)$.

Future works

(1) Question: Are there algebraic structures on $(\mathbb{V}_R, \sigma_R)$?
Answer: Let R be a range set such that $R \setminus \{0\}$ is a (multiplicative) subgroup of $(0, \infty)$. For every prime p, there exists a valued field structure on $(\mathbb{V}_R, \sigma_R)$ such that it is a valued field extension of $\mathbb{Q}_p$.

(2) Is the hyperspace $(\mathcal{K}(\mathbb{U}), \mathcal{HD}_\rho)$ isometric to $(\mathbb{U}, \rho)$?
I think this is negative.

(3) Is the hyperspace of the random graph is isometric to the random graph?
The **random graph (Rado graph, Erdős–Rényi graph)** is the Urysohn type universal space for $\mathcal{F}(\{0, 1, 2\})$ (all $\{0, 1, 2\}$-valued finite met. sps).
I think this is negative.

(4) Clarify (all) properties of $\mathrm{Isom}(\mathbb{V}_R, \sigma_R)$.

My papers

(1) Yoshito Ishiki, **Constructions of Urysohn universal ultrametric spaces**, to appear in p-Adic Numbr. Ultr. Anal. Appl, arXiv:2302.00305.

(2) Yoshito Ishiki, **Uniqueness and homogeneity of non-separable Urysohn universal ultrametric spaces**, Topology and its Applications, 342 (2024), Paper No. 108762.

(3) Yoshito Ishiki, **Characterizations of Urysohn universal ultrametric spaces**, arXiv:2303.1747.

(4) Yoshito Ishiki, **A non-Archimedean Arens–Eells isometric embedding theorem on valued fields**, arXiv:2309.06704v1.

(5) Yoshito Ishiki, **Algebraic structures on non-Archimedean Urysohn universal metric spaces**, in preparation.

Thank you very much for your attention.