Strongly rigid metrics in spaces of metrics

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MSJ Spring Meeting 2024
2024/Mar/19

Space of metrics

Definition 1

For a metrizable space X, we denote by $\mathrm{Met}(X)$ the set of all metrics on X that generate the same topology of X.
We define a metric $\mathcal{D}_X$ on $\mathrm{Met}(X)$ by

$$\mathcal{D}_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|.$$

In what follows, we consider that $\mathrm{Met}(X)$ is equipped with the topology induced by $\mathcal{D}_X$.

The speaker have studied the topological properties of $\mathrm{Met}(X)$. For example, for a certain property $\mathcal{P}$ on metric spaces, the speaker proved that the denseness and determined the Borel hierarchy of $\{\, d \in \mathrm{Met}(X) \mid (X, d) \text{ satisfies } \mathcal{P} \,\}$.

Strongly rigid metrics

Definition 2

A metric space (X, d) is **rigid** if every isometric bijection $f: X \to X$ must be the identity map 1_X.

Definition 3

A metric d on a set X is said to be **strongly rigid** if for all $x, y, u, v \in X$, the relations $d(x, y) = d(u, v)$ and $d(x, y) \neq 0$ imply that $\{x, y\} = \{u, v\}$. Equivalently, the map $f_d : [X]^2 \to [0, \infty)$ defined by $f_d(\{x, y\}) = d(x, y)$ is injective.

If $3 \leq \mathrm{Card}(X)$, "Strongly rigid" $\Longrightarrow$ "Rigid".

In 1972, Janos proved that:

Theorem A

A separable metrizable space X with clopen basis ($\mathrm{ind}(X) = 0$) has a strongly rigid metric $d \in \mathsf{Met}(X)$.

Theorem B (Janos, Martin)

The following statements are true.

(1) *If a metrizable space X admits a strongly rigid metric $d \in \mathrm{Met}(X)$, then $\mathrm{Card}(X) \leq \mathfrak{c}$ and X has small inductive dimension 0 ($\mathrm{ind}(X) = 0$).*

(2) *If a metrizable space X has large inductive dimension 0 ($\mathrm{Ind}(X) = 0$, equivalently $\dim(X) = 0$) and $\mathrm{Card}(X) \leq \mathfrak{c}$, then X has a strongly rigid metric $d \in \mathrm{Met}(X)$.*

Remark

(1) (Open problem): It is unknown that whether X with $\mathrm{Card}(X) \leq \mathfrak{c}$ having small inductive dimension 0 ($\mathrm{ind}(X) = 0$) admits a strongly rigid metric $d \in \mathrm{Met}(X)$.

In this work, the speaker proved that the set of all strongly rigid metrics is dense in $\mathrm{Met}(X)$.

Main results

The definition of strongly rigid metrics seems to be linear independence.

Definition 4

For a metrizable space X, we denote $\mathrm{LI}(X)$ the set of all $d \in \mathrm{Met}(X)$ such that if $x, y, u, v \in X$ satisfies $x \neq y$, $u \neq v$, and $\{x, y\} \neq \{u, v\}$, then the positive real numbers $d(x, y)$ and $d(u, v)$ are linearly independent over $\mathbb{Q}$.

Theorem 1 ([I])

Let X be a metrizable space with $\mathrm{Card}(X) \leq \mathfrak{c}$ having large inductive dimension 0. Let $\epsilon \in (0, \infty)$ and $d \in \mathrm{Met}(X)$. Then there exists $e \in \mathrm{LI}(X)$ such that $\mathcal{D}_X(d, e) \leq \epsilon$. Namely, the set $\mathrm{LI}(X)$ is dense in $(\mathrm{Met}(X), \mathcal{D}_X)$. Moreover, if X is completely metrizable, we can choose e as a complete metric.

Definition 5

For a metrizable space X, we denote by $\mathrm{SR}(X)$ the set of all strongly rigid metrics in $\mathrm{Met}(X)$.

Theorem 2 ([I])

Let X be a metrizable space with $\mathrm{Card}(X) \leq \mathfrak{c}$ having large inductive dimension 0. Then the set $\mathrm{SR}(X)$ is dense in $\mathrm{Met}(X)$. Moreover, if X is σ-compact, then $\mathrm{SR}(X)$ is dense G_δ in $(\mathrm{Met}(X), \mathcal{D}_X)$.

Remark

Theorem 2 is an analogue of Rouyer's result stating that the set of all strongly rigid compact metric spaces is comeager in the Gromov–Hausdorff space.

Definition 6

For a metrizable space X, we denote by $\mathrm{R}(X)$ the set of all rigid metrics in $\mathrm{Met}(X)$.

Theorem 3 ([I])

Let X be a metrizable space having large inductive dimension 0. If X is σ-compact and satisfies $3 \leq \mathrm{Card}(X) \leq \mathfrak{c}$, then the set $\mathrm{R}(X)$ is comeager in $(\mathrm{Met}(X), \mathcal{D}_X)$.

The concept of **comeager-ness** is used in the theory of Baire category, and it is a topological analogue of co-null sets (complements of null sets) in measure theory.

Remark

Theorem 3 is a 0-dimensional analogue of the result that the set of all rigid Riemannian (or pseudo- Riemannian) metrics is open dense in the space of Riemannian metrics with the Whitney C^∞-topology

As an application, we can obtain a strange metric on a 0-dimensional space.

Theorem 4 ([I])

Let X be a second-countable locally compact Hausdorff space having large inductive dimension 0. Then there exists $d \in \mathrm{Met}(X)$ such that for every $\xi \in X$, the map $F_{\xi}\colon X \to [0, \infty)$ defined by $F_{\xi}(x) = d(x, \xi)$ is a topological embedding. Moreover, we can choose d as a complete metric.

Future Work

Question

Let X be a metrizable space. **We do not assume that X has large inductive dimension 0.** Is it true that $\mathrm{R}(X)$ is dense in $\mathrm{Met}(X)$? If X is σ-compact, is $\mathrm{R}(X)$ comeager in $\mathrm{Met}(X)$?

Question

Let X be a metrizable space. Is $(\mathrm{Met}(X), \mathcal{D}_X)$ Baire?
It is proven by the speaker that if X is second-countable locally compact metrizable space, then $(\mathrm{Met}(X), \mathcal{D}_X)$ is completely metrizable, especially, Baire.

My paper

(1) Yoshito Ishiki, *Strongly rigid metrics in spaces of metrics*, Topology Proceedings, 63 (2024), 125–148.

Thank you very much for your attention.