ROUGH SKETCH OF THEORIES OF RETRACTS

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Abstract. Borsuk's book "Theory of retracts" and Hu's book "theory of retracs" are often chosen as major books on retracts and references containing fundamental theory and properties of AR, ANR, AE, and ANE. However, these books are out of print for a long time. So, in this memo, I exhibit the main results on the theory of retracts, along with relevant references, in preparation for future use.

Contents

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1. Introduction

In this memo, whenever we consider a class of topological spaces, it is closed under taking closed subsets.

In the theory of retracts, we consider extending continuous maps, so we prefer to think topological spaces admitting ample continuous maps, continuous function. A representative one is the class of Tychonoff spaces (completely regular spaces), in which distinct points are functionally separated and topologies are generated by all real-valued functions. In this reason, we consider only subclass of the class of Tychonoff spaces. This manuscript mainly treat the following classes of topological spaces, and note that these are closed under taking closed subsets.

(1) "Tychonoff.": the class of all Tychonoff spaces.
(2) "Nor.": the class of all normal spaces. In this memo, we impose T_1.
(3) "Her.Nor.": the class of all hereditary normal spaces, spaces all which subspaces are normal. It is also called T_5.
(4) "Per.Nor.": the class of all perfectly normal spaces, spaces all which closed subsets are G_δ. It is also called T_6.
(5) "Coll.Nor.": the class of all collectionwise normal spaces.
(6) "κ-Clec.Nor.": the class of all κ-collectionwise normal spaces for a cardinal κ.
(7) "ParaCpt.": the class of all paracompact Hausdorff spaces.
(8) "κ-ParaCpt." the class of all κ-paracompact normal spaces for a cardinal κ.
(9) "Lind.": the class of all Lindelöf spaces.
(10) "Mono.Nor.": the class of all monotonically normal spaces.
(11) "Euc.": the class of all spaces homeomorphic to closed subset of Euclidean spaces. This class coincides with the class of all locally compact finite-dimensional separable metrizable spaces, which is a consequence of dimension theory.
(12) "Cpt.": the class of all compact Hausdorff spaces.
(13) "Met'ble.": the class of all metrizable spaces.
(14) "Sep.Met'ble.": the class of all separable metrizable spaces.
(15) "Cpt.Met'ble.": the class of all compact metrizable spaces.

For fundamental concepts in general topology such as paracompactness, collectionwise normality, see [73], [80].

The construction of adjunction spaces plays an important role in theory of retracts for equivalence of ANR and ANE for certain classes.

Theorem 1.1. *The following properties are stable under the construction of adjunction spaces.*

(1) *"Nor."*
(2) *"Her.Nor."*
(3) *"Per.Nor."*

(4) *"Coll.Nor."*
(5) *"κ-Clec.Nor."*
(6) *"ParaCpt."*
(7) *"κ-ParaCpt."*
(8) *"Mono.Nor."*
(9) *"Cpt."*

Proof. See [26], [54], [60], [78]. □

Theorem 1.2. *The following classes are stable under weak topology (weak topology in the sense of Whitehead and Morita)*

(1) *"Nor."*
(2) *"Per.Nor."*
(3) *"Coll.Nor."*
(4) *for each cardinal κ, the class "κ-Clec.Nor."*
(5) *"ParaCpt."*
(6) *for each cardinal κ, the class "κ-ParaCpt."*
(7) *(*[60, Theorem 2.2]*) "Mono.Nor."*

Proof. see [61] and [62], [63]. □

Lemma 1.3 (The tube lemma)**.** *Let X be a topological space, K be a compact space, $x \in X$ and U be an open set of $X \times K$. If U contains $\{x\} \times K$, then there exists a neighborhood N of x such that $N \times K \subset U$.*

Proof. Since K is compact and U is open, we can take finite open neighborhoods $O_1, \dots, O_n$ of x and open subsets $V_1, \dots, V_n$ such that

$$\{x\} \times K \subset \bigcup_{i=1}^{n} O_i \times V_i \subset U. \tag{1.1}$$

Puting $N = \bigcap_{i=1}^{n} O_i$ finishes the proof. □

The following theorem and lemma are presented in [36] in a generalized form. The decomposition in the next theorem is known as the Whitney–Dugundji decomposition (see [15]).

Theorem 1.4. *Let (X, w) be a metric space, and A be a closed subset of X. Then there exists a a quadruplet*

$$\mathcal{WD} = [\mathcal{O}, \{\mathbf{y}_O\}_{O\in\mathcal{O}}, \{\mathbf{a}_O\}_{O\in\mathcal{O}}, \{\varphi_O\}_{O\in\mathcal{O}}],$$

such that the following conditions are satisfied:

(WD1) *the family $\mathcal{O}$ is a locally finite open covering of $X \setminus A$ consisting of open sets of $X \setminus A$, each $\mathbf{y}_O$ is a point in $X \setminus A$, each $\mathbf{a}_O$ is a point in A, and each φ_O is a continuous function from X to* $[0, 1]$*;*

(WD2) *For every $O \in \mathcal{O}$, we have*

$$O \subset U\left(\mathbf{y}_O, \frac{w(\mathbf{y}_O, A)}{4^2}; w\right);$$

(WD3) *For every $O \in \mathcal{O}$, we have*

$$w(\mathbf{y}_O, \mathbf{a}_O) \le \frac{4^2+1}{4^2} w(\mathbf{y}_O, A);$$

(WD4) *For every $O \in \mathcal{O}$, we have*

$$\operatorname{supp}(\varphi_O) \subseteq O,$$

where $\operatorname{supp}(\varphi_O)$ *stands for the support of* φ_O *defined as the set* $\{ x \in X \mid \varphi_O(x) \neq 0 \}$*;*

(WD5) *The family* $\{\varphi_O\}_{O\in\mathcal{O}}$ *satisfies*

$$\sum_{O\in\mathcal{O}} \varphi_O(x) = \begin{cases} 0 & \text{if } x \in A; \\ 1 & \text{if } x \in X \setminus A. \end{cases}$$

(WD6) *if $x \in X \setminus A$ and $O \in \mathcal{O}$ satisfy $\varphi_O(x) > 0$, then*

$$\operatorname{diam}_w(O \cup \{\mathbf{y}_O\}) \le \cdot w(x, A).$$

(WD7) *Moreover, if M is separable and has covering dimension n, then $\mathcal{O}$ can be choose so that its covering degree is $\le n$.*

Proof. For each $i \in \mathbb{Z}$, we define an open set V_i of $X \setminus A$ by

$$V_i = \{ x \in X \setminus A \mid 2^{i-1} < w(x, A) < 2^{i+1} \}.$$

Note that if $2 \le |i - j|$, then $V_i \cap V_j = \emptyset$. Put

$$\mathcal{V}_i = \{ V_i \cap U\left(x, w(x, A)4^{-2}; w\right) \mid x \in V_i \}$$

Then each $\mathcal{V}_i$ is an open covering of V_i. Put $\mathcal{W} = \bigcup_{i\in\mathbb{Z}_{\ge 0}} \mathcal{V}_i$. Since X is paracompact (see [76, Corollary 1] and [70]), we can take a locally finite partition of unity $\{\psi_a\}_{a\in I}$ subordinated to $\mathcal{W}$ (see [55, Proposition 2]). By taking finite sums if necessary, we may assume that for every distinct pair $a, b \in I$, we have $\operatorname{supp}(\psi_a) \neq \operatorname{supp}(\psi_b)$. Put $\mathcal{O} = \{ \operatorname{supp}(\psi_a) \mid a \in I \}$ and $\varphi_O = \psi_a$, where ψ_a is a unique member such that $\operatorname{supp}(\psi_a) = O$. Then, the family $\mathcal{O}$ is a locally finite open covering of $X \setminus A$, and it refines $\mathcal{W}$.

For each $O \in \mathcal{O}$, choose $\mathbf{y}_O \in X \setminus A$ so that

$$O \subset U(\mathbf{y}_O, w(\mathbf{y}_O, A)4^{-2}; w).$$

Note that if $O \subseteq V_i$, then there exists $j \in \mathbb{Z}$ such that $|i - j| \le 1$ and $\mathbf{y}_O \in V_j$. We also choose $\mathbf{a}_O \in A$ so that we have

$$w(\mathbf{y}_O, \mathbf{a}_O) \le \frac{4^2+1}{4^2} w(\mathbf{y}_O, A).$$

In this setting, if $\varphi_O(x) > 0$ and $O \in \mathcal{V}_i$, then there exists $j \in \mathbb{Z}$ with $|i - j| \le 1$ such that we have $x \in V_j \cap U(\mathbf{y}_O, w(\mathbf{y}_O, A)4^{-k-1}; w)$ and $\mathbf{y}_O \in V_j$. By the construction, we observe that the quadruplet

$$[\mathcal{O}, \{\mathbf{y}_O\}_{O\in\mathcal{O}}, \{\mathbf{a}_O\}_{O\in\mathcal{O}}, \{\varphi_O\}_{O\in\mathcal{O}}]$$

is a (k, w)-WD collection.

We next verify that the quadruplet satisfies the condition (WD6). Assume that $x \in X \setminus A$ and $O \in \mathcal{O}$ satisfy $\varphi_O(x) > 0$. Take $i \in \mathbb{Z}$ such that $O \in \mathcal{V}_i$, and take $j \in \mathbb{Z}$ such that $\mathbf{y}_O \in V_j$. Then we have $|i - j| \leq 1$. Under this setting, we obtain

$$\begin{aligned}&\operatorname{diam}_w(O \cup \{\mathbf{y}_O\}) \leq \operatorname{diam}_w U(\mathbf{y}_O, w(\mathbf{y}_O, A)4^{-2}; w)\\&\leq 2 \cdot 4^{-2} w(\mathbf{y}_O, A) \leq 2 \cdot 4^{-2} \cdot 2^{j+1} \leq 2 \cdot 4^{-2} \cdot 2^{i+2}\\&= 2^4 \cdot 4^{-2} 2^{i-1} \leq 2^4 \cdot 4^{-2} w(x, A) = \cdot w(x, A)\end{aligned}$$

To prove (WD7), using finite-dimensionness and separability of we choose a locally finite open covering $\mathcal{P}$ such that its degree is $\leq n$ and it refines $\mathcal{W}$. Take a locally finite partition of unity and conduct the same method above. Then (WD7) is satisfied. This finishes the proof. □

Lemma 1.5. *Under the same assumptions as in Theorem 1.4. Let* $\mathcal{WD} = [\mathcal{O}, \{\mathbf{y}_O\}_{O \in \mathcal{O}}, \{\mathbf{a}_O\}_{O \in \mathcal{O}}, \{\varphi_O\}_{O \in \mathcal{O}}]$ *be a a quadruplet stated in Theorem 1.4. Then, for every* $a \in A$ *and for every* $x \in X$*, if* $\varphi_O(x) > 0$*, then we have*

$$w(a, \mathbf{a}_O) \leq 4w(a, x).$$

Proof. Our proof is based on that of [4, Statement 2.3]. Since $\varphi_O(x) > 0$, the conditions (WD4) and (WD2) implies that $x \in O$ and

$$O \subset U\left(\mathbf{y}_O, \frac{w(\mathbf{y}_O, A)}{4^2}\right).$$

Thus we have

$$w(x, \mathbf{y}_O) \leq \frac{w(\mathbf{y}_O, A)}{4^2}.$$

Using this estimation and the condition (WD3), we also have

$$\begin{aligned}w(a, \mathbf{a}_O) &\leq w(a, x) + w(x, \mathbf{y}_O) + w(\mathbf{y}_O, \mathbf{a}_O)\\&\leq w(a, x) + \frac{1}{4^2} w(\mathbf{y}_O, A) + \frac{1 + 4^2}{4^2} w(\mathbf{y}_O, A)\\&= w(a, x) + \frac{4^2 + 2}{4^2} w(\mathbf{y}_O, A).\end{aligned}$$

Namely,

$$w(a, \mathbf{a}_O) \leq w(a, x) + \frac{4^2 + 2}{4^2} w(\mathbf{y}_O, A). \tag{1.2}$$

Moreover, by

$$w(\mathbf{y}_O, A) \leq w(\mathbf{y}_O, a) \leq w(\mathbf{y}_O, x) + w(x, a) \leq \frac{1}{4^2} w(\mathbf{y}_O, A) + w(x, a),$$

we obtain

$$\left(1 - \frac{1}{4^2}\right) w(\mathbf{y}_O, A) \leq w(x, a),$$

and hence, we also obtain

$$w(\mathbf{y}_O, A) \leq \frac{4^2}{4^2-1} w(x,a). \tag{1.3}$$

Therefore, combining these inequalities (1.2) and (1.3), we confirm the following computations.

$$\begin{aligned} w(a, \mathbf{a}_O) &\leq w(a,x) + \frac{4^2+2}{4^2} w(\mathbf{y}_O, A) \leq w(a,x) + \frac{4^2+2}{4^2-1} w(a,x) \\ &= \frac{2\cdot 4^2+1}{4^2-1} w(a,x) = \frac{2+4^{-2}}{1-4^{-2}} w(x,a) \leq \frac{2+1}{3/4} w(x,a) = 4w(x,a). \end{aligned}$$

This finishes the proof. □

2. AR, ANR, AE, and ANE:Definitions

2.1. **Definitions and notations.** Let X be a topological space, and E and A be subsets of X with $A \subset E$. We say that A is a *retract* of E if there exists a continuous map $r\colon E \to A$ such that $r(x) = x$ for all $x \in A$. The map r mentioned above is called a *retraction*.

Definition 2.1. Let $\mathcal{C}$ be a class of topological spaces. A space X in $\mathcal{C}$ is an *absolute retract for* $\mathcal{C}$ *(*AR($\mathcal{C}$)*)* if X belongs to $\mathcal{C}$ and for every space Y in $\mathcal{C}$, and for every closed embedding $l\colon X \to Y$, the (closed) subset $l(X)$ of Y is a retract of Y. Namely, AR is a space that is always a retract whenever it is embedded into a closed subset of a metrizable space.

Definition 2.2. Let $\mathcal{C}$ be a class of topological spaces. A space X in $\mathcal{C}$ is an *absolute neighborhood retract for* $\mathcal{C}$ *(*ANR($\mathcal{C}$)*)* if X belongs to $\mathcal{C}$ and for every space Y in $\mathcal{C}$, and for every closed embedding $l\colon X \to Y$, the (closed) subset $l(X)$ of Y is a retract of a open subset U of Y with $l(X) \subset U$. Namely, we can take a retraction $r\colon U \to l(X)$. In other words, the subset $l(X)$ has a neighborhood that has a retraction into $l(X)$.

Definition 2.3. Let $\mathcal{C}$ be a class of topological spaces. A topological space X is an *absolute extensor for* $\mathcal{C}$ *(*AE($\mathcal{C}$)*)* if for every space Z in $\mathcal{C}$, for every closed subset A of Z, and for every continuous map $f\colon A \to Z$, there exists a continuous map $F\colon Z \to X$ such that $F|_A = f$. In this definition, we do not assume $X \in \mathcal{C}$.

Definition 2.4. Let $\mathcal{C}$ be a class of topological spaces. A topological space X is an *absolute neighborhood extensor for* $\mathcal{C}$ *(*ANE($\mathcal{C}$)*)* if for every space Z in $\mathcal{C}$, for every closed subset A of Z, and for every continuous map $f\colon A \to Z$, there exist an open subset U with $A \subset U$ and a continuous map $F\colon U \to X$ such that $F|_A = f$. Namely, the target set X makes the domain set A have a neighborhood U that enable f to extend to U continuously. In this definition, we do not assume $X \in \mathcal{C}$.

3. Abstract theory

In this section, we mainly cite Hanner's paper [26].

3.1. **Basic theorems.**

Theorem 3.1. *Let $\mathcal{C}$ is a class of a topological space. Then the following are true:*

(1) $\mathrm{AR}(\mathcal{C}) \subset \mathrm{ANR}(\mathcal{C})$.
(2) $\mathrm{AE}(\mathcal{C}) \subset \mathrm{ANE}(\mathcal{C})$.
(3) $\mathcal{C} \cap \mathrm{AE}(\mathcal{C}) \subset \mathrm{AR}(\mathcal{C})$.
(4) $\mathcal{C} \cap \mathrm{ANE}(\mathcal{C}) \subset \mathrm{ANR}(\mathcal{C})$.
(5) *Every retract of a space in* $\mathrm{AE}(\mathcal{C})$ *belongs* $\mathrm{AE}(\mathcal{C})$.
(6) *Every neighborhood retract of a space in* $\mathrm{AE}(\mathcal{C}) \cup \mathrm{ANE}(\mathcal{C})$ *belongs* $\mathrm{ANE}(\mathcal{C})$.

Proof. See [26, p.318]. □

Proposition 3.2. *Let $\mathcal{C}$ be a class of topological spaces. Then every open subset of X in* $\mathrm{ANE}(\mathcal{C})$ *belongs* $\mathrm{ANE}(\mathcal{C})$ *too.*

Proof. See [25, Lemma 3.1]. Let O be an open subset of X. For $Z \in \mathcal{C}$ and a closed subset A of Z, take a continuous map $f\colon A \to O$. The taget set of f can be change to X and using $X \in \mathrm{ANE}(\mathcal{C})$, we can find $\phi\colon V \to X$, where V is a neighborhood of A. Putting $U = \phi^{-1}(O)$ and $F = \phi|_U$ completes the proof. □

Theorem 3.3. *Let $\mathcal{C}$ be a class of topological space. Assume that either of the following is true:*

(Adjunct) *the class $\mathcal{C}$ is stable under adjunction spaces; namely, if $P, Q \in \mathcal{C}$, A be a closed subset of P and $f\colon A \to Q$ be a continuous map, then we have $P \cup_f Q \in \mathcal{C}$;*

(Convex) *we have $\mathcal{C} \subset$ "Met'ble." and for every $W \in \mathcal{C}$, there exists a convex space C of a locally convex space such that W can be embedded into C as a closed subset and $C \in \mathcal{C}$.*

Then we have the following two statements.

(C1) $\mathcal{C} \cap \mathrm{ANE}(\mathcal{C}) = \mathrm{ANR}(\mathcal{C})$.
(C2) $\mathcal{C} \cap \mathrm{AE}(\mathcal{C}) = \mathrm{AR}(\mathcal{C})$.

Proof. See [24] for an archetype of the proof with (Adjunct). See [26, Theorems 8.1 and 8.2].

Now we assume (Adjunct) and prove (C1). The case (C2) is similar. The inclusion $\mathcal{C} \cap \mathrm{ANE}(\mathcal{C}) \subset \mathrm{ANR}(\mathcal{C})$ is already known. Take $X \in \mathrm{ANR}(\mathcal{C})$. For any continuous map $f\colon A \to X$, where A is closed in $Z \in \mathcal{C}$. Since X is closed in $Z \cup_f X$ and $Z \cup_f X \in \mathcal{C}$, there exists a neighborhood O of X in $Z \cup_f X$ and a retraction $r\colon O \to X$. Then there exists an open subset U with $A \subset U$ such that $k(U) \subset O$, where $k\colon Z \to Z \cup_f X$. In this situation, the map $F = r \circ k\colon U \to X$ becomes an extension of f.

Next we assume (Convex) and prove (C1). Take $X \in \mathrm{ANR}(\mathcal{C})$, and a convex set C of a locally convex linear space E with $C \in \mathcal{C}$. Let $r\colon U \to X$, where U is a neighborhood of X in C. be a retraction. For any $Z \in \mathcal{C}$, any closed subset A of Z, and any continuous map $f\colon A \to X$, due to the Dugundji extension theorem 4.2, there exists $\phi\colon Z \to C$ with $\phi|_A = f$. Putting $O = \phi^{-1}(U)$ and $F = r\circ\phi|_O\colon O \to X$ proves $X \in \mathrm{ANE}(\mathcal{C})$, and hence (C1).

Obviously, the class "Cpt." satisfies the (Adjunct). It is clear that "Euc." satisfies (Convex)

Since a closed convex hull of a compact set in a complete metrizable locally convex space is compact [2, Theorem 5.35] (see also [74]), using some embedding theorem into linear spaces, we see that "Cpt.Met'ble." satisfies the (Convex).

For "$\aleph_0$-ParaCpt.", see [33].

For "κ-ParaCpt.", see [47, Corollary 1 and Theorem 2]

For "Mono.Nor.", see [60, Theorem 1.1 and Corollary 1.2].

For hereditary properties, see [34], and [54].

For hypernomality, see [35] (the hypernomality seems to be equivalent to the extreme disconnectedness). □

Remark 3.1. See also Theorem 1.1.

3.2. **Cartesian product.**

Proposition 3.4. *Let $\mathcal{C}$ be a class of topological spaces. Let $\{X_i\}_{i\in I} \subset$ $\mathrm{AE}(\mathcal{C})$. Then the Cartesian product $\prod_{i\in I} X_i$ is* $\mathrm{AE}(\mathcal{C})$.

Proof. See [26, p.318]. □

Proposition 3.5. *Let $\mathcal{C}$ be a class of topological spaces. Let $\{X_i\}_{i\in I} \subset$ $\mathrm{ANE}(\mathcal{C})$ and I be a finite index set. Then the Cartesian product $\prod_{i\in I} X_i$ is* $\mathrm{ANE}(\mathcal{C})$.

Proof. See [26, p.318]. □

3.3. **Union-stability.**

Proposition 3.6. *Let $\mathcal{C}$ be a class of topological spaces with $\mathcal{C} \subset$ "Nor.". Let X be a topological space. If there exist $\mathrm{ANE}(\mathcal{C})$ open subsets U_1 and U_2 with $X = U_1 \cup U_2$, then X is $\mathrm{ANE}(\mathcal{C})$. As a corollary, a topological space X that is a finite union of open $\mathrm{ANE}(\mathcal{C})$, the space X is $\mathrm{ANE}(\mathcal{C})$.*

Proof. See [25, Theorem 3.3] and [26, Theorem 23.1]. Take $Z \in \mathcal{C}$ and let A be a closed set of Z, and $f\colon A \to X$ a continuous map. Using normality of Z, take disjoint open subsets V_1 and V_2 such that $f^{-1}(U_1 \setminus U_2) \subset V_1$ and $f^{-1}(U_2 \setminus U_1) \subset V_2$. Since $f(A \setminus (V_1 \cup V_2)) \subset U_1 \cap U_2$, and $U_1 \cap U_2$ is $\mathrm{ANE}(\mathcal{C})$ (Proposition 3.2), we can take a closed neighborhood L of $A \setminus (V_1 \cup V_2)$ and a continuous map $g_0\colon L \to U_1 \cap U_2$ that is an extension of $f|_{A\setminus(V_1\cup V_2)}$. Define $s_1\colon ((V_1\cup A)\setminus V_2)\cup(L\setminus(V_1\cup V_2))\to U_1$ and $s_2\colon ((V_2\cup A)\setminus V_1)\cup(L\setminus(V_1\cup V_2))\to U_2$ by

$$s_1(z)=\begin{cases} g_0(x) & \text{if } x\in(L\setminus(V_1\cup V_2));\\ f(x) & \text{if } x\in A\setminus V_2.\end{cases} \tag{3.1}$$

$$s_2(z)=\begin{cases} g_0(x) & \text{if } x\in(L\setminus(V_1\cup V_2));\\ f(x) & \text{if } x\in A\setminus V_1.\end{cases} \tag{3.2}$$

By

$$\begin{aligned}&((V_2\cup A)\setminus V_1)\cap(L\setminus(V_1\cup V_2))\\ &=((V_1\cup A)\setminus V_2)\cap(L\setminus(V_1\cup V_2))=A\setminus(V_1\cup V_2),\end{aligned}$$

the maps s_1 and s_2 are continuous. Moreover, since the intersection of $(A\setminus V_2)\cup(L\setminus(V_1\cup V_2))$ and $(A\setminus V_1)\cup(L\setminus(V_1\cup V_2))$ is equal to $L\setminus(V_1\cup V_2)$, we can glue s_1 and s_1 together, and we obtain a continuous map $F\colon V_1\cup L\cup V_2\to U_1\cup U_2$. In this situation, the set $V_1\cup L\cup V_2$ is a neighborhood of A. Therefore, we conclude that $X=U_1\cup U_2\in \mathrm{ANE}(\mathcal{C})$. □

Remark 3.2. The closed sum version of Proposition 3.6 seems to be difficult. See [26, Example 23.4], [41], [40], and [71].

Theorem 3.7. *Let $\mathcal{C}$ be a topological space with $\mathcal{C}\subset$ "Coll.Nor.". Let X be a paracompact Hausdorff space. If X is locally* $\mathrm{ANE}(\mathcal{C})$*, then it is* $\mathrm{ANE}(\mathcal{C})$.

Proof. See [57, Proposition 4.1] and [25, pp.393–394]. We omit the proof here. □

Theorem 3.8. *Let $\mathcal{C}$ be a class of topological spaces with $\mathcal{C}\subset$ "Nor.". If X is Lindelöf normal and locally* $\mathrm{ANE}(\mathcal{C})$*, then $X\in\mathrm{ANE}(\mathcal{C})$.*

Proof. See [26, Theorem 19.4]. □

3.4. **Contractibility.** A topological space X is *contractible* if X is homotopic to a point in X. Namely, there exists a homotopy $h\colon X\times[0,1]\to X$ such that h_0 is the identity map and h_1 takes a value of a single point.

In this memo, a topological space X is *locally contractible in the feeble sense* if every point $x\in X$ has a neighborhood system in which every member U has a homotopy $h\colon U\times[0,1]\to X$ such that h_0 is the inclusion map $U\to X$ and the image of h_1 is a single point. Namely, with parameter $t\in[0,1]$, the set U changes to a single point and U can go outside of U.

A topological space X is *locally contractible* if every point $x\in X$ has a neighborhood system in which every member U contains an open neighborhood V of x and there exists a homotopy $h\colon V\times[0,1]\to U$

such that h_0 is the inclusion map $V \to U$ and the image of h_1 is a single point.

For the complexity of the concept of the local contractibility, see [52], and [28].

Theorem 3.9. *Let $\mathcal{C}$ be a class of topological spaces with $\mathcal{C} \subset$ "Nor.". Then every contractible space X in* $\mathrm{ANE}(\mathcal{C})$ *is* $\mathrm{AE}(\mathcal{C})$.

Proof. See [26, Theorem 12.3]. Take $Z \in \mathcal{C}$ and let A be a closed subset of Z and $f\colon A \to X$ be a continuous map. Since X is $\mathrm{ANE}(\mathcal{C})$, there exist a neighborhood O of A and a continuous map $g\colon O \to X$ that is an extension of f. Take a homotopy $h\colon X \times [0,1] \to X$ such that h_0 is the identity map and h_1 is a constant map taking a value $p \in X$. Using the normality of Z, we take a open set W such that $A \subset W$ and $\mathrm{CL}(W) \subset O$, and a continuous map $u\colon Z \to [0,1]$ such that $u(A) = \{1\}$ and $u(Z \setminus W) = \{0\}$. Define a map $F\colon X \to X$ by

$$F(x) = \begin{cases} h(f(x), u(x)) & \text{if } x \in O; \\ p & \text{if } x \notin \mathrm{CL}(W) \end{cases} \tag{3.3}$$

In this setting, for all $x \in O \cap (Z \setminus \mathrm{CL}(W))$, we have $h(f(x), u(x)) = p$. Thus, the map F is well-defined and continuous on Z. By definition, we also observe that $F|_A = f$, and hence X is $\mathrm{AE}(\mathcal{C})$. ☐

Theorem 3.10. *Let $\mathcal{C}$ be a class of topological spaces, and X be a space with $X \times [0,1] \in \mathcal{C}$. Then we have:*

(1) *If X is* $\mathrm{AE}(\mathcal{C})$*, then X is contractible.*
(2) *If X is regular and $X \in$* $\mathrm{ANE}(\mathcal{C})$*, then X is locally contractible.*

Proof. See [26, Theorem 12.4] and [26, Theorem 28.1]. This theorem is a reason why mathematician defined and considered the concept of countable paracompactness. First we show (1). Choose $p \in X$ and define a continuous map $h\colon X \times \{0\} \cup X \times \{1\} \to X$ by

$$h(x) = \begin{cases} x & \text{if } x \in X \times \{0\}; \\ p & \text{if } x \in X \times \{1\}. \end{cases} \tag{3.4}$$

Then h is define on a closed subset of $X \times [0,1] \in \mathcal{C}$. Since $X \in \mathrm{ANE}(\mathcal{C})$, we can take $H\colon X \times [0,1] \to X$ that is an extension of h. Then H is a witness of the contractibility of X.

We next demonstrate (2). Take an arbitrary point $p \in X$ and an arbitrary open neighborhood N of p. Using a naive method similar to the above proof of (1), we obtain the local contractibility in the feeble sense. Now, by $X \in \mathrm{ANE}(\mathcal{C})$ and Proposition 3.2, the set N is $\mathrm{ANE}(\mathcal{C})$. Using regularity of X, we can take a closed neighborhood L of x with $L \subset N$. Since $\mathcal{C}$ is closed under taking closed subsets, we see that $L \times [0,1] \in \mathcal{C}$. Define a map $h\colon L \times \{0\} \cup \{p\} \times [0,1] \cup L \times \{1\} \to N$

by

$$h(x,t) = \begin{cases} x & \text{if } (x,t) \in L \times \{0\}; \\ p & \text{if } (x,t) \in \{p\} \times [0,1]; \\ p & \text{if } (x,t) \in X \times \{1\}. \end{cases} \tag{3.5}$$

Then h is continuous and defined on a closed subset of $X \times [0,1]$. Since $N \in \mathrm{ANR}(\mathcal{C})$, there exist a neighborhood W of the domain h and a continuous map $g\colon W \to N$ that is an extension of h. Using compactness of $[0,1]$, and $\{p\} \times [0,1] \subset W$, by the tube lemma (Lemma 1.3), we can take a neighborhood V of p such that $V \times [0,1] \subset W$. Put $H = g|_{V\times[0,1]}\colon V \times [0,1] \to N$. Then H and V are witness of the local contractibility of X. $\square$

Theorem 3.11. *Let $\mathcal{C}$ be a class of topological spaces with $\mathcal{C} \subset$ "Nor.", and Z be a space with $Z \times [0,1] \in \mathcal{C}$. Then Z is* AE($\mathcal{C}$) *if and only if Z is contractible and* ANE($\mathcal{C}$).

Proof. See [26, Theorem 12.6]. For "κ-ParaCpt.", see [47, Theorem 3].

If Z is AE($\mathcal{C}$), then Theorem 3.10 implies that Z is contractible.

Next, assume that Z is contractible ANE($\mathcal{C}$). In this case, Theorem 3.9 shows that Z is AE($\mathcal{C}$). $\square$

3.5. **Homotopy.** Let $\mathcal{C}$ be a class of topological spaces. We say that *the homotopy extension theorem holds for $\mathcal{C}$ and X* if for every $Z \in \mathcal{C}$ and for every closed subset A of Z, and for every continuous map $f\colon Z \times \{0\} \cup (A \times [0,1]) \to X$, there exists a continuous map $F\colon Z \times \{0\} \times [0,1] \to X$ that is an extension of f. It seems to be a dual concept of the homotopy extension property.

Theorem 3.12 (Borsuk's homotopy extension theorem)**.** *Let $\mathcal{C}$ be a class of topological spaces such that every $Z \in \mathcal{C}$ satisfies $Z \times [0,1] \in \mathcal{C}$ and $\mathcal{C} \subset$ "Nor.". If $E \in$* ANE($\mathcal{C}$)*, then the homotopy extension theorem holds for $\mathcal{C}$ and E.*

Proof. See [26, Theorem 28.2]. See also [30, p.86]. Let $Z \in \mathcal{C}$, A be a closed subset of Z, and $Z\colon X \cup \{0\} \cup A \times [0,1] \to E$ be a continuous map. Since $Z \times [0,1] \in \mathcal{C}$ and $E \in \mathrm{ANE}(\mathcal{C})$, we obtain an open neighborhood O of the domain of h in $Z \times [0,1]$ and a continuous map $\widetilde{h}\colon O \to E$. Using Lemma 1.3, we can find an open subset W of Z such that $A \subset W$ and $W \times [0,1] \subset O$. Using the normality of Z, we obtain a continuous map $\phi\colon Z \to [0,1]$ such that $\phi(A) = \{1\}$ and $\phi(Z \setminus W) = \{0\}$. Define $H\colon X \times [0,1] \to E$ by $H(x,t) = \widetilde{h}(x, t \cdot \phi(x))$. Then H is a continuous map that is an extension of h. $\square$

Theorem 3.13. *Let $\mathcal{C}$ be a class of topological spaces such that every $Z \in \mathcal{C}$ satisfies $Z \times [0,1] \in \mathcal{C}$ and $\mathcal{C} \subset$ "Nor.". Then X is locally* ANE($\mathcal{C}$) *if and only if every $x \in X$ has a neighborhood V such that for every $Z \in \mathcal{C}$, for every closed subset A of Z, and for every continuous* *map $f\colon A \to V$, there exists a continuous map $F\colon Z \to X$ such that $F(x) = f(x)$ for all $x \in A$.*

Proof. See [26, Theorem 28.3] and [25, Theorem 5.1]. we first assume that X is locally ANE($\mathcal{C}$). Take $Z \in \mathcal{C}$, a closed subset A of Z. For every $x \in X$, take an open neighborhood N that is ANE($\mathcal{C}$). By Theorem 3.10, we may assume that there exists a homotopy $h\colon V \times [0,1] \to X$ such that h_0 is the identity map and h_1 is a constant map taking a value $p \in X$ (in this proof, we only use the local contractibility in the feeble sense). Take any continuous map $f\colon A \to V$. Define $w\colon Z \times \{0\} \cup A \times [0,1] \to X$ by $w(z,t) = h(f(z), 1-t)$. Note that since h_0 is constant, the map w can be defined on $Z \times \{0\}$. Then w is continuous. Theorem 3.12 gives us a continuous map $H\colon Z \times [0,1] \to X$. Put $F(x) = H(z,1)$, then it is an extension of f.

Next we assume that X satisfies the latter condition in the statement. Considering $h = F|_{F^{-1}(V)}\colon F^{-1}(V) \to X$ completes proof. □

Corollary 3.14. *Let $\mathcal{C}$ be a class of topological spaces such that every $Z \in \mathcal{C}$ satisfies $Z \times [0,1] \in \mathcal{C}$ and $\mathcal{C} \subset$ "Nor.". If X is locally contractible in the feeble sense and the homotopy extension theorem holds for $\mathcal{C}$ and X, then X is locally* ANE($\mathcal{C}$).

Proof. See [26, Theorem 28.4]. A careful examination of the proof of Theorem 3.13 yields the theorem. □

Remark 3.3. Borsuk's original paper is [9]. For a generalization of Borsuk's homotopy extension theorem, see, for example, [64] and [66]. In [75], For a normal space X, and a closed subset A, the space $X \times \{0\} \cup A \times [0,1]$ is C^*-embedded. As a result, for every class $\mathcal{C}$ of topological space, if E is a compact ANE($\mathcal{C}$), then the homotopy extension theorem holds for $\mathcal{C}$ and E.

3.6. **Relationships between retracts and extensors on different classes.** There are so many theorems, we omit the proofs of theorems in this subsection.

Theorem 3.15. *The following statements are true:*

1. *Every Hausdorff X in* ANE(*"Tychonoff."*) *is a one-point space.*
2. $\mathbb{R} \in$ ANR(*"Tychonoff."*)
3. $\mathbb{R} \notin$ AR(*"Tychonoff."*)

Proof. See [26, Example 17.4]. □

Theorem 3.16. *The following statements are true:*

1. *A metrizable space X is* ANR(*"Nor."*) *if and only if X is separable,* ANR(*"Met'ble."*)*, and is absolutely G_δ.*
2. *A metrizable space X is* AR(*"Nor."*) *if and only if X is separable,* AR(*"Met'ble."*)*, and is absolutely G_δ.*

Proof. See [24, Theorems 4.1 and 4.2] and [14, Theorem 1]. □

Theorem 3.17. *The following statements are true:*

(1) *A metrizable space X is* ANR(*"Coll.Nor."*) *if and only if X is* ANR(*"Met'ble."*), *and is absolutely G_δ.*
(2) *A metrizable space X is* AR(*"Coll.Nor."*) *if and only if X is* AR(*"Met'ble."*), *and is absolutely G_δ.*

Proof. See [24, Theorems 4.1 and 4.2] and [14, Theorem 1]. □

Theorem 3.18. *A metrizable space X is*

$$\mathrm{ANR}(\text{“Coll.Nor.”} \cap \text{“Per.Nor.”}) \tag{3.6}$$

if and only if X is ANR(*"Met'ble."*).

Proof. [14, Corollary 2]. □

Theorem 3.19. *The following statements are true:*

(1) ANR(*"Tychonoff."*) $\subset$ ANE(*"Nor."*).
(2) AR(*"Tychonoff."*) $\subset$ AE(*"Nor."*).

Proof. See [26, Theorems 11.1–11.2]. □

Theorem 3.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of topological spaces such that $\mathcal{C} \subset$ "Nor." and $\mathcal{D} \subset$ "Nor.", all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ satisfy $A \times [0,1] \in \mathcal{C}$ and $B \times [0,1] \in \mathcal{D}$. Let X be a an* ANE($\mathcal{C}$). *Then X is* AR($\mathcal{D}$) *if and only if* ANR($\mathcal{D}$).

Proof. See [26, Theorem 13.1]. In this setting, Theorem 3.10 implies that X is contractible. So, Theorem 3.11 gives the desired equivalence. This finishes the proof. □

Theorem 3.21. *The following are true:*

(1) ANR(*"Cpt.Met'ble."*) $\subset$ ANR(*"Nor."*)
(2) ANR(*"Cpt."*) $\subset$ ANR(*"Nor."*)
(3) ANR(*"Sep.Met'ble."*) $\subset$ ANR(*"Met'ble."*)

Proof. See [26, Theorems 13.2–13.4]. □

Theorem 3.22. *A space X in "Met'ble." $\cap$* ANR(*"ParaCpt."*) *is absolutely G_δ.*

Proof. See [26, Theorem 14.1]. □

Theorem 3.23. *Let X be in* ANE(*"Nor."*). *Then X is c.c.c.*

Proof. See [26, Theorem 14.4]. Bing's example G is essential for this theorem. □

Theorem 3.24. *Let X be in "ParaCpt." $\cap$* ANE(*"Nor."*). *Then X is Lindelöf.*

Proof. See [26, Theorem 14.5]. □

Theorem 3.25. *If X is* ANR(*"Sep.Met'ble."*) *and absolutely G_δ, then X is* ANR(*"Nor."*).

Proof. See [26, Theorem 14.5]. □

Theorem 3.26. *The following statements are true:*

(1) *If X is metrizable and* ANR(*"Tychonoff."*)*, then X is separable and locally compact.*
(2) *"Met'ble."* $\cap$ AR(*"Tychonoff."*) $\subset$ *"Cpt."*
(3) AR(*"Cpt."*) $\subset$ AR(*"Tychonoff."*)
(4) ANR(*"Cpt."*) $\subset$ ANR(*"Tychonoff."*)
(5) AR(*"Cpt.Met'ble."*) $\subset$ AR(*"Tychonoff."*)
(6) *If X is* ANR(*"Met'ble."*)*, and separable, locally compact, the $X \in$* ANR(*"Tychonoff."*)

Proof. See [26, Theorem 15.4–16.6]. See [29, p.1053] □

Theorem 3.27. *Let X be an* ANR(*"Met'ble."*)*. Then the following are true:*

(1) *If X is* ANR(*"ParaCpt."*)*, then X is absolutely G_δ.*
(2) *If X is absolutely G_δ, then X is* ANR(*"Coll.Nor."*)
(3) *X is* ANR(*"Tychonoff."*) *is and only if X is separable and locally compact.*

Proof. See [26, Theorem 17.1]. □

Theorem 3.28. *Let X be an* AR(*"Met'ble."*)*. Then the following are true:*

(1) *If X is* AR(*"ParaCpt."*)*, then X is absolutely G_δ.*
(2) *If X is absolutely G_δ, then X is* AR(*"Coll.Nor."*)
(3) *X is* AR(*"Tychonoff."*) *is and only if X is compact.*

Proof. See [26, Theorem 17.2]. □

Theorem 3.29. *Let Z be a separable metrizable space. Then the following are true:*

(1) *Z is* ANR(*"Met'ble."*) *if and only if it is* ANR(*"Sep.Met'ble."*).
(2) *Z is* AR(*"Met'ble."*) *if and only if it is* AR(*"Sep.Met'ble."*).

Proof. See [29, Theorem 3.1] □

Theorem 3.30. *Let Y be a metrizable space that is* ANE(*"Met'ble."*)*. Then the following are true:*

(1) *Z is* ANE(*"Per.Nor."* $\cap$ *"ParaCpt."*).
(2) *Z is* ANE(*"ParaCpt."*)*. if and only if Z is absolutely G_δ.*
(3) *Z is* ANE(*"Per.Nor."*) *if and only if Z is separable and absolute G_δ.*
(4) *Z is* ANE(*"Tychonoff."*) *if and only if Z is empty or a one-point space.*

Proof. See [56, Theorem 3.1]. □

Theorem 3.31. *Let Y be a metrizable space that is* AE(*"Met'ble."*)*. Then the following are true:*

(1) *Z is* AE(*"Per.Nor."* $\cap$ *"ParaCpt."*).
(2) *Z is* AE(*"ParaCpt."*) *if and only if Z is absolutely G_δ.*
(3) *Z is* AE(*"Per.Nor."*) *if and only if Z is separable and absolute G_δ.*
(4) *Z is* AE(*"Tychonoff."*) *if and only if Z is empty or a one-point space.*

Proof. See [56, Theorem 3.1]. □

Theorem 3.32. *Let Y be a metrizable space that is* ANR(*"Met'ble."*). *Then the following are true:*

(1) *Z is ANR for paracompact spaces containing Z as a G_δ set.*
(2) *Z is* ANR(*"ParaCpt."*) *if and only if Z is absolutely G_δ.*
(3) *Z is* ANR(*"Per.Nor."*) *if and only if Z is separable.*
(4) *Z is* ANR(*"Nor."*) *if and only if Z is separable and absolute G_δ.*
(5) *Z is* ANR(*"Tychonoff."*) *if and only if Z compact.*

Proof. See [56, Theorem 3.2]. □

Theorem 3.33. *Let Y be a metrizable space that is* AR(*"Met'ble."*). *Then the following are true:*

(1) *Z is AR for paracompact spaces containing Z as a G_δ set.*
(2) *Z is* AR(*"ParaCpt."*) *if and only if Z is absolutely G_δ.*
(3) *Z is* AR(*"Per.Nor."*) *if and only if Z is separable.*
(4) *Z is* AR(*"Nor."*) *if and only if Z is separable and absolute G_δ.*
(5) *Z is* AR(*"Tychonoff."*) *if and only if Z locally compact and separable.*

Proof. See [56, Theorem 3.2]. □

Theorem 3.34. *If X is metrizable and* ANE(*"Her.Nor."*), *then X is absolutely G_δ.*

Proof. See [31]. □

Theorem 3.35. *Let X be a separable metric space.*

(1) *X is* ANR(*"$\aleph_0$-ParaCpt."*) *if and only if X is* ANR(*"Sep.Met'ble."*) *and absolutely G_δ.*
(2) *X is* ANR(*"$\aleph_0$-ParaCpt."*) *if and only if X is* ANR(*"Nor."*).

Proof. See [32, Theorems 3 and 4]. □

4. Examples

4.1. Extension theorems.

Theorem 4.1 (Tietze–Urysohn's extension theorem)**.** *The following are true:*

(1) *Every interval of the real line is* AE(*"Nor."*).

(2) $\mathbb{R}^n \in \mathrm{AE}($*"Nor."*$)$ *for all* $n \in \mathbb{Z}_{\geq 0}$.
(3) $\mathbb{R}^\kappa \in \mathrm{AE}($*"Nor."*$)$ *for all cardinal* κ.

Theorem 4.2 (Dugundji's extension theorem)**.** *Every convex subset of a locally convex linear topological space is* AE(*"Met'ble."*).

Proof. See [15], and [4]. □

Theorem 4.3 (Dugundji)**.** *Let* κ *be a cardinal. Let* E *be an arbitrary Banach space with weight* $\leq \kappa$. *Then we have*

$$B \in \mathrm{AE}(\text{“}\kappa\text{-Clec.Nor.”}). \tag{4.1}$$

Namely, for every κ*-collectionwise normal space* T_1 X, *for every closed subset* A *of* X, *and for every continuous function* $f\colon A \to B$, *there exists* $F\colon X \to B$.

Proof. See [15]. See also [32, Theorem 1] and [69]. □

4.2. Embedding theorems.

Theorem 4.4 (Arens–Eells embedding theorem)**.** *For every metric space* (X, d), *there exist a normed space* $(E, \|{*}\|$, *and a map* $I\colon X \to E$ *such that*

(1) *for all* $x, y \in X$, *we have* $\|I(x) - I(y)\| = d(x, y)$;
(2) *the set* $I(X)$ *is closed in* E;
(3) *the set* $I(X)$ *is linearly independent; namely, every finite distinct members in* $I(X)$ *is linearly independent.*

Proof. See [5], [77, footnote (1)], and [58]. □

4.3. Zero-dimensional spaces.

Theorem 4.5. *(There is no theory of retracts on ultrametrizable spaces). Let* X *be an ultrametrizable space. Then every non-empty closed subset* A *is a retract of* X.

Proof. See [17, Theorem 9.1], [42, Theorem 4], [18, Lemma], and [11, Theorem 2.9]. □

5. Structures of combinatorics(Under construction)

5.1. Simplicial complex. For applications of simplicial complexes, see [13]. See also [37].

Theorem 5.1. *Every compact metric* ANR(*"Met'ble."*) *is homotopic to compact simplicial complex.*

Proof. See [79]. □

Theorem 5.2. *Every locally finite connected simplicial complex is an* ANR(*"Sep.Met'ble."*).

Proof. See [25, Corollary 3.5]. □

5.2. **CW complex.** We mainly refer to [45].

Theorem 5.3. *Every CW complex is paracompact and perfectly normal.*

Proof. Theorems 1.1 and 1.2 implies the theorem. □

Theorem 5.4 (Dugundji)**.** *Every simplicial complex with CW topology is an* ANE(*"Met'ble."*)*.*

Proof. See [16] and [40, Corollary 2]. □

The paper [22] gives a characterization of AR("Met'ble.") and ANR("Met'ble.") in terms of CW complexes.

6. ENR

6.1. **Basic statements.** We mainly refer to [10, Appendix E] and [12, Chapter IV §8]. In this memo, we say that a space Z *is ENR* if Z can be topologically embedded into a euclidean space and every image of such embedding is a neighborhood retract. In this definition, we do not impose the closed-ness of images.

Lemma 6.1. *Let Z be a Hausdorff space, and S be a locally compact space. Then there exist a closed subset F and an open subset G of Z such that $S = F \cap G$.*

Proof. This lemma is a consequence of a classical argument of the local closedness. For each $x \in S$, take a compact neighborhood L_x and an open subset O_x in Z with $x \in O_x$ and $O_x \cap S \subset L_x$. Put $G = \bigcup_{x\in X} O_x$. Now let us show S is a closed subset of G. Take $z \in G \setminus S$. Then there exists $x \in S$ with $z \in O_x$. Since $z \notin S$ and $O_x \cap S \subset L_x$, we have $z \notin L_x$. Namely $z \in O_x \setminus L_x$ and $(O_x \cap L_x) \cap S = \emptyset$. From the fact that Z is Hausdorff and L_x is compact, we see that $O_x \setminus L_x$ is open in Z. Therefore $G \setminus S$ is open in G. Hence there exists a closed subset F of Z such that $S = F \cap G$. □

Lemma 6.2. *Let $n \in \mathbb{Z}_{\geq 1}$ and X be a subset of $\mathbb{R}^n$ which is a neighborhood retract. Then there exist a closed subset F and an open subset G of $\mathbb{R}^n$ such that $X = F \cap G$. As a result, the space X is locally compact, and it can be embedded into $\mathbb{R}^{n+1}$ as a closed subset.*

Proof. See [10, Appendix E]. Since X is a retract of some neighborhood O of X, X is a closed subset of O. Thus there exists a closed subset F of $\mathbb{R}^n$ such that $X = F \cap G$. We omit the proofs of the latter parts (Use proper functions). □

Lemma 6.3. *Let Z be an* ANR(*"Euc."*)*. Then Z is ENR.*

Proof. Take any topological embedding $h\colon Z \to h(Z) \subset \mathbb{R}^n$, and put $Q = h(Z)$. By Lemma 6.2, the space Z is locally compact, and hence so is Q. Thus, we can find a closed subset F and an open subset G in

$\mathbb{R}^n$ such that $Q = G \cap F$ (Lemma 6.1). Note that Q is a closed subset of G. Since Z is ANE("Euc."), and since $G \in$ "Euc.", we obtain a neighborhood W of Q in G and a continuous map $g\colon W \to Z$ such that $g|_Q = h^{-1}$. Define $r = h \circ g\colon W \to Q$. Then r is a retraction of Q. □

Lemma 6.4. *Let Z be a space. Then the following are equivalent to each other:*

(1) *Z is ENR;*
(2) *Z is in* ANR(*"Euc."*)*;*

Proof. See [10, Appendix E]. Proof of (1) $\Longrightarrow$ (2): It suffices to show that an ENR Z can be embedded into a Euclidean space as a closed subset. This is a consequence of Lemma 6.2.

Proof of (2) $\Longrightarrow$ (1): Lemma 6.3. □

Theorem 6.5. *Every ENR is* ANR(*"Met'ble."*) *and* ANR(*"Nor."*)*.*

Proof. This theorem is proven by Dugundji's extension theorem (Theorem 4.2) and Tietze–Urysohn theorem (Theorem 4.1), and the fact that every ENR can be embedded into a euclidean space as a closed subset. □

Corollary 6.6. *Every contractible ENR is* AR(*"Met'ble."*)*.*

Proposition 6.7. *Let X be a paracompact Hausdorff space. If X is locally* ENR*, then X is* ENR*. As a result, every σ-compact connected manifold is* ENR*.*

Proof. We give another proof here. Let M be a manifold stated in the proposition. Since M is locally Euclidea, it is also locally ANE("Euc.") (consider closed balls). As a result of standard arguments in the theory of manifolds, the space M is paracompact. Thus M is ANE("Euc.") (Theorem 3.7), and hence it is ANR("Euc.") by Theorem 3.3. □

Theorem 6.8. *Every compact ENR is a retract of a finite simplicial complex.*

Proof. Assume $X \subset \mathbb{R}^n$ and X is a compact ENR. Take an open neighborhood U of X. We may assume that U is bounded. Then $\{U, \mathbb{R}^n \setminus X\}$ becomes a covering of $\mathbb{R}^n$. Since X is compact, the set $\{U, \mathbb{S}^{n+1} \setminus X\}$ is also a covering of $\mathbb{S}^{n+1}$. Take any metric d on $\mathbb{S}^{n+1}$. By the compactness of $\mathbb{S}^{n+1}$, there exists a Lebesgue number $\delta > 0$ of the covering $U, \mathbb{S}^{n+1} \setminus X$. Take a simplicial subdivision K of $\mathbb{S}^{n+1}$ which mesh is smaller than δ. Let L be the sum of all simplex in K contained in U. Then L becomes a subcomplex of K and $L \subset U$. Since X is a retraction of U, it is also retraction of L. □

6.2. **Local contractibility and Euclidean spaces.**

Lemma 6.9. *Let $\mathcal{C}$ be a class of topological spaces. Let Z be a space. Assume that for every $x \in Z$, for every spaceM $\in \mathcal{C}$, and for every closed subset A of M, there exists V such that every continuous map $f\colon A \to V$ extends to $g\colon M \to X$. Then Z is locally* ANE($\mathcal{C}$).

Proof. Considering $h = g|_{g^{-1}(V)}\colon g^{-1}(V) \to Z$ completes proof. □

Let $\mathbb{S}^n$ and $\mathbb{B}^n$ denote the n-dimensional sphere and the n-dimensional closed ball, respectively.

Lemma 6.10. *Let Z be a locally contractible space. Then every x has a neighborhood system such that all member U has V with $x \in V \subset U$ satisfying that for every $n \in \mathbb{Z}_{\geq 0}$, every continuous map $f\colon \mathbb{S}^n \to V$ can be extended to a continuous map $F\colon \mathbb{B}^{n+1} \to U$*

Proof. The cone of $\mathbb{S}^n$ is homeomorphic to $\mathbb{B}^{n+1}$. □

Theorem 6.11. *Every locally contractive space is locally* ANE(*"Euc."*).

Proof. See [46], [44], and [43]. Let Z be a locally contractive space. Now we prove that for every $n \in \mathbb{Z}_{\geq 0}$ for every $x \in Z$, for every closed subset A of $\mathbb{R}^n$, there exists a neighborhood H of x such that every continuous map $f\colon A \to H$ extends to $g\colon \mathbb{R}^n \to X$.

Using Lemma 6.10 repeatedly, we obtain $H = N_0 \subset N_1 \subset \cdots \subset N_n$ such that for every $i = 0, \dots, n-1$, and for every $k \in \mathbb{Z}_{\geq 0}$, any continuous map $q\colon \mathbb{S}^k \to N_i$ is extended to a continuous map $\tilde{q}\colon \mathbb{B}^{k+1} \to N_{i+1}$. Let $f\colon A \to O$ be a continuous map. Applying Theorem 1.4 to $\mathbb{R}^n$ and A, and take

$$\mathcal{WD} = [\mathcal{O}, \{\mathbf{y}_O\}_{O\in\mathcal{O}}, \{\mathbf{a}_O\}_{O\in\mathcal{O}}, \{\varphi_O\}_{O\in\mathcal{O}}],$$

satisfying the conditions (WD1)–(WD7). Let P be a nerve generated by $\mathcal{O}$ with the weak topology. Put $E = A \cup P$ and for every $W \subset \mathbb{R}^n$, put

$$\widetilde{W} = (W \cap A) \cup \bigcup \left\{ \mathrm{St}(O) \mid O \in P^{(0)}(= \mathcal{O}), O \subset W \right\}. \tag{6.1}$$

We define a topology $E = A \cup P$ by generating the open sets of P and all $\widetilde{W}$'s for open subsets W of $\mathbb{R}^n$. In this setting, using $\{\varphi_O\}_{O\in\mathcal{O}}$, define $\Phi\colon \mathbb{R}^n \to E$ by

$$\Phi(x) = \begin{cases} x & \text{if } x \in A; \\ O \mapsto \varphi_O(x) & \text{if } x \in \mathbb{R}^n \setminus A. \end{cases} \tag{6.2}$$

Then Φ is continuous. Note that P has dimension $\leq n$ as a simplicial complex. By induction, we define maps $u_i\colon A \cup P^{(i)} \to N_i$ for $i = 0, \dots, n$ with $u_i|_A = f$. If $i = 0$, we deffine $u_0(O) = f(\mathbf{a}_O)$. Lemma 1.5 implies that u_0 is continuous. When we define $u_i\colon A \cup P^{(i)} \to N_i$, then Lemma 6.10 enables us to take a map $u_{i+1}\colon A \cup P^{(i+1)} \to N_{i+1}$. The continuity follows from (WD6). Then we obtain continuous map $u_n\colon E \to X$ with $u_n|_A = f$. Consequently, putting $g = u_n \circ \Phi\colon \mathbb{R}^n \to X$ and Lemma 6.9 imply X is locally ANR("Euc."). □

Combining above arguments, we obtain the following:

Theorem 6.12. *Let Z be a space. Then the following are equivalent to each other:*

(1) *Z is ENR;*
(2) *Z is in* ANR(*"Euc."*)*;*
(3) *Z is a locally compact locally contractible finite-dimensional separable metrizable space.*

Proof. See [10, Appendix E] and [12, Proposition 8.12]. Lemma 6.4 states that (1) $\iff$ (2).

Proof of (1) $\Longrightarrow$ (3): This implication is a corollary of Lemma 6.2 and Theorem 3.10.

Proof of (3) $\Longrightarrow$ (2): As a consequence of dimension theory, Z can be embedded into a euclidean space as a closed subset. It suffices to show that it is ANE("Euc."). Combining the paracompactness of Z, Theorems 6.11, 3.7, and 3.3, we see that Z is ANR("Euc."). This finishes the proof. $\square$

Theorem 6.13. *Every finite CW complex is ENR.*

Proof. See [10, CorollaryE. 8]. Slightly difficult. $\square$

7. Specialization to metric spaces(Under construction)

See [23, Appendix A] and [73].

Proposition 7.1. *The following statements are true.*

(1) *Every* AR(*"Met'ble."*) *is contractive;*
(2) *Every* ANR(*"Met'ble."*) *is locally contractive.*

Proposition 7.2. *An* ANE(*"Met'ble."*) *is* AE(*"Met'ble."*) *if and only if it is contractible.*

8. Other Branches of the theory of retracts

General references of ANR: [26], [82].

n-dimensional ANR: [53].

Homological properties of ANR: [39].

The paper [6] thinks the sufficient conditions for the quotient space of ANR becomes ANR.

ANR theory in infinite-dimensional topology, see [7].

Uniform ANR: [20], and [59].

The paper [49] contains history of the theory of retracts. The paper [38] is a survey on modifications of ANR's.

G-ANR: [67], and [3].

For the work of Kiyoshi Iséki, see [27].

Retracts on stratifiable spaces: [8].

Generalizations of absolute retracts: [50], [1], [81], [19], [68], [72], amd [51].

Shape theory: [21], [65], and [48].

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