DIMENSION THEORY OF SEPARABLE METRIZABLE SPACES

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Abstract. This manuscript was translated into English using Codex. The purpose of this manuscript is to present fundamental results in dimension theory for separable metrizable spaces. We focus on three classical dimensions: the small inductive dimension, the large inductive dimension, and the covering dimension. Among other things, we prove that these three dimensions coincide, establish the existence of embeddings into Euclidean spaces, and determine the dimension of Euclidean space.

To give an accessible proof of the coincidence theorem, we introduce a notion called the decomposition dimension, based on decompositions into zero-dimensional spaces. For separable metrizable spaces, this dimension also agrees with the three classical dimensions above.

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1. Introduction

1.1. **Background.** In this manuscript we develop the topological dimension theory of separable metrizable spaces, starting from zero-dimensional spaces. It is difficult to give a single answer to the question of what dimension is. Roughly speaking, dimension is an integer measuring the degrees of freedom of a space, but many different notions of dimension are now used for a wide variety of spaces. In the setting of topological spaces, dimension theory may be viewed as a refinement of the classification of spaces by connectedness. It measures how difficult a space is to separate, for example by the boundaries of neighborhoods (the small and large inductive dimensions) or by the orders of refinements of covers (the covering dimension). These dimensions may differ in general, but one of the fundamental theorems of dimension theory states that they all coincide for separable metrizable spaces.

A distinctive feature of this manuscript is the introduction of the *decomposition dimension*, which is based on decompositions into zero-dimensional spaces and is intended to make the coincidence theorem easier to understand. For separable metrizable spaces, the decomposition dimension also agrees with the three dimensions above.

We aim to cover the broad content of the classical text [5] and to make results from older sources accessible. More specifically, we discuss the coincidence of the three dimensions (Theorem 2.29), the determination of the dimension of Euclidean space (Theorem 3.6), the embedding of finite-dimensional spaces into Euclidean spaces (Theorem 6.8), and the invariance of domain theorem (Theorem 4.2). We also include an embedding theorem for manifolds (Theorem 7.7), which is useful for modern people.

For more advanced topics in dimension theory, see [4]. The book [8] treats settings more general than separable metrizable spaces. A recent and readily available reference is [2].

We now describe the organization of the manuscript. The remainder of Section 1 reviews the facts from general topology used in dimension theory and introduces our notation for special spaces such as spheres.

In Section 2 we develop dimension theory for separable metrizable spaces. We first introduce the decomposition dimension, the small and large inductive dimensions, and the covering dimension, and then establish the basic properties of the decomposition dimension. The countable sum theorem is particularly important. In fact, our definition of the decomposition dimension was inspired by the proof of the countable sum theorem for the small inductive dimension. It measures the minimum number of zero-dimensional subspaces whose union is the whole space. It has long been known that this quantity agrees with the small inductive dimension for separable metrizable spaces. The zero-dimensional subspaces in its definition need not be open, closed, Borel, or even Baire sets. It is therefore noteworthy that dimension can be recovered from the number of pieces in such a potentially wild decomposition.

Our main strategy for comparing dimension functions is to construct a base whose members have boundaries of dimension one less than that of the ambient space. With the exception of the upper estimate for the covering dimension, all comparisons between dimension functions are obtained in this way. One advantage of introducing the decomposition dimension is that it yields a direct proof of Theorem 2.21, which bounds the covering dimension above by an inductive dimension.

Section 3 proves Brouwer's fixed-point theorem. This theorem is needed to determine the dimension of Euclidean space, since, roughly speaking, Brouwer's fixed-point theorem is equivalent to

$$n - 1 < \dim\mathbb{R}^n.$$

The sections beginning with Section 4 contain applications of dimension theory and may be read in any order because they are independently of each other. In Section 4, we prove the invariance of domain theorem: the image of an open subset of n-dimensional Euclidean space under a continuous injection into n-dimensional Euclidean space is open.

In Section 5, we prove that an n-dimensional subspace of n-dimensional Euclidean space has nonempty interior. This result is better regarded as an application of the countable dense homogeneity of Euclidean space than as a direct application of dimension theory.

In Section 6, we prove that every n-dimensional space embeds into $(2n + 1)$-dimensional Euclidean space and discuss universal spaces for n-dimensional spaces.

Finally, in Section 7, we use only the finiteness of the decomposition dimension of a manifold to construct a differentiable embedding of the manifold into a Euclidean space.

1.2. **Basic facts from general topology.** Whenever we assume a separation axiom such as normality, we also assume the T_1 axiom.

Lemma 1.1 (Tube lemma)**.** *Let X be a topological space, K be a compact space, $x \in X$, and let U be an open subset of $X \times K$. If $\{x\} \times K \subset U$, then there exists a neighborhood N of x such that $N \times K \subset U$.*

Proof. Since K is compact and U is open, there exist open neighborhoods $O_1, \dots, O_n$ of x and open subsets $V_1, \dots, V_n$ of K such that

$$\{x\} \times K \subset \bigcup_{i=1}^{n} O_i \times V_i \subset U. \tag{1.1}$$

Taking $N = \bigcap_{i=1}^{n} O_i$ completes the proof. □

Lemma 1.2 (The pasting lemma)**.** *Let X and Y be topological spaces. Assume that $A_1, \dots, A_m$ are closed subset of X with $X = \bigcup_{i=1}^{m} A_i$, and for each $i \in \{1, \dots, m\}$, $f_i \colon A_i \to Y$ is a continuous map with $f_i(a) = f_j(a)$ for all $a \in A_i \cap A_j$ and for all $i, j \in \{1, \dots, m\}$. Define $f \colon X \to Y$ by $f(x) = f_i(x)$ if $x \in A_i$. Then f is well-defined and continuous on X.*

Proof. By the assumption that $f_i(a) = f_j(a)$ for all $a \in A_i \cap A_j$ and $i, j \in \{1, \dots, m\}$, the map f is well-defined. For every closed subset C of Y, we have $f^{-1}(C) = \bigcup_{i=1}^{m} f_i^{-1}(C)$. Since each f_i is continuous and m is finite, we conclude that $f^{-1}(C)$ is closed. Thus f is continuous on X. This lemma can be regarded as a consequence of the fact that every topological space has a final topology induced from finite closed subsets and inclusion maps. □

Theorem 1.3 (Urysohn metrization theorem)**.** *Let X be a second-countable normal space. Then X admits a topological embedding into $[0,1]^{\aleph_0}$. In particular, the space X is metrizable.*

Proof. We omit the proof. □

Theorem 1.4 (Tietze–Urysohn extension theorem)**.** *Let X be a normal space, A be a closed subset of X, and let $f \colon A \to [0,1]$ be continuous. Then there exists a continuous map $F \colon X \to [0,1]$ such that $F|_A = f$.*

Definition 1.1 (Open shrinking)**.** Let $\mathcal{U}$ be an open cover of a topological space X. A family

$$\{ V_U \mid U \in \mathcal{U} \}$$

is called an **open shrinking** of $\mathcal{U}$ if it is an open cover of X and

$$\mathrm{CL}(V_U) \subset U \tag{1.2}$$

for every $U \in \mathcal{U}$.

Lemma 1.5 (Shrinking lemma for covers)**.** *Let $\{U_1, \dots, U_m\}$ be a finite open cover of a normal space X. Then there exists an open cover $\{V_1, \dots, V_m\}$ such that $\mathrm{CL}\, V_i \subset U_i$ for every $i \in \{1, \dots, m\}$. Equivalently, every finite open cover of a normal space has an open shrinking.*

Proof. We argue by induction. For each $l \in \{1, \dots, m\}$, we construct $V_1, \dots, V_l$ satisfying the following conditions.

(1) $\mathrm{CL}\, V_i \subset U_i$ for every $i \in \{1, \dots, l\}$;
(2) $\{V_1, \dots, V_l, U_{l+1}, \dots, U_m\}$ covers X.

Assume that the construction has been completed through $l = k$. Define

$$A = \left(\bigcap_{i=1}^{k} X \setminus V_i \right) \cap \left(\bigcap_{i=k+1}^{m} X \setminus U_i \right). \tag{1.3}$$

Then A is closed, and condition (2) implies that $A \subset U_{k+1}$. By normality, there exists an open set V_{k+1} such that

$$A \subset V_{k+1} \subset \mathrm{CL}\, V_{k+1} \subset U_{k+1}.$$

This gives conditions (1) and (2) for $l = k + 1$. The case $l = m$ completes the proof. □

We next introduce a notion of similarity for families of sets.

Definition 1.2. Let I be an index set. Two families $\mathcal{A} = \{A_i\}_{i \in I}$ and $\mathcal{B} = \{B_i\}_{i \in I}$ of subsets of a set X are called **similar** if, for every $S \subset I$, we have

$$\bigcap_{s \in S} A_s = \emptyset \iff \bigcap_{s \in S} B_s = \emptyset.$$

Equivalently, one intersection is nonempty exactly when the other is.

In topology one is often interested in constructing similar families subject to inclusions such as $B_i \subset A_i$, with the sets A_i and B_i required to be open or closed. We restrict ourselves to the finite case needed below.

Lemma 1.6. *Let X be a normal space, and let $\{F_1, \dots, F_m\}$ be a family of closed sets and $\{U_1, \dots, U_m\}$ a family of open sets such that $F_i \subset U_i$ for every $i \in \mathbb{Z}_{\geq 0}$. We do not assume that either family covers X. Then there exists a family of open sets $\{W_1, \dots, W_m\}$ such that we have $F_i \subset W_i$ and $\subset \mathrm{CL}\, W_i \subset U_i$ for every $i \in \{1, \dots, m\}$, and the families $\{F_1, \dots, F_m\}$ and $\{\mathrm{CL}\, W_1, \dots, \mathrm{CL}\, W_m\}$ are similar.*

Proof. We argue by induction. For each $l \in \{1, \dots, m\}$, construct sets $W_1, \dots, W_l$ satisfying:

(1) $\{\mathrm{CL}\, W_1, \ldots, \mathrm{CL}\, W_l\} \cup \{F_{l+1}, \ldots, F_m\}$ is similar to $\{F_1, \ldots, F_m\}$;
(2) $F_j \subset W_j$ and $\mathrm{CL}\, W_j \subset U_j$ for every $j \in \{1, \ldots, l\}$.

Assume that the construction has been completed through $l = k$. Let $\mathcal{K}$ be the family of all intersections of finitely many members of

$$\{\mathrm{CL}\, W_1, \ldots, \mathrm{CL}\, W_k\} \cup \{F_{k+1}, \ldots, F_m\}$$

that are disjoint from F_{k+1}. Put $H = X \setminus \bigcup \mathcal{K}$. Since $\mathcal{K}$ is finite, the set H is open, and $F_{k+1} \subset H$. By normality, choose an open set W_{k+1} such that

$$F_{k+1} \subset W_{k+1} \subset \mathrm{CL}\, W_{k+1} \subset H \cap U_{k+1}.$$

The definition of $\mathcal{K}$ shows that

$$\{\mathrm{CL}\, W_1, \ldots, \mathrm{CL}\, W_k\} \cup \{F_{k+1}, \ldots, F_m\}$$

and

$$\{\mathrm{CL}\, W_1, \ldots, \mathrm{CL}\, W_{k+1}\} \cup \{F_{k+2}, \ldots, F_m\}$$

are similar. The inductive hypothesis therefore gives the desired similarity with $\{F_1, \ldots, F_m\}$. Taking $l = m$ completes the proof. $\square$

Lemma 1.7. *Let X be a separable metrizable space, and let $\mathcal{B}$ be an open base of X. Then $\mathcal{B}$ contains a countable subfamily $\mathcal{A}$ that is itself an open base base of X.*

Proof. Every subspace of a separable metrizable space is separable and metrizable, and hence Lindelöf. Choose a countable base $\mathcal{C}$ for X. For each $U \in \mathcal{C}$, there is a countable family $\mathcal{A}_U \subset \mathcal{B}$ such that

$$\bigcup_{V \in \mathcal{A}_U} V = U.$$

Define

$$\mathcal{A} = \bigcup_{U \in \mathcal{C}} \mathcal{A}_U.$$

Then $\mathcal{A}$ is the required countable base. $\square$

Definition 1.3. Let X be a topological space and $S \subset X$. We denote the boundary of S in X by $\mathrm{Bdry}_X S$. Thus $x \in \mathrm{Bdry}_X S$ if and only if every neighborhood H of x meets both S and $X \setminus S$. In general, we see that

$$\mathrm{Bdry}_X S = \mathrm{CL}_X S \setminus \mathrm{Int}_X S,$$

where CL_X and Int_X denote closure and interior in X, respectively. When the ambient space is clear, we simply write $\mathrm{Bdry}\, S$, $\mathrm{CL}\, S$, and $\mathrm{Int}\, S$.

Lemma 1.8. *Let X be a topological space and $A \subset S \subset X$. Then we have*

$$\mathrm{Bdry}_S A \subset S \cap \mathrm{Bdry}_X A.$$

Proof. Put $O = \operatorname{Int}_X A$ and $C = \operatorname{CL}_X A$. Then $O \cap S$ and $C \cap S$ are respectively open and closed in S, and

$$O \cap S \subset A \subset C \cap S.$$

Consequently,

$$\operatorname{Bdry}_S A \subset (C \cap S) \setminus (O \cap S).$$

Moreover, $(C \cap S) \setminus (O \cap S) = (C \setminus O) \cap S = (\operatorname{CL}_X A \setminus \operatorname{Int}_X A) \cap S = S \cap \operatorname{Bdry}_X A$. This proves the assertion. □

Lemma 1.9. *Let X be a topological space and $N \subset A \subset X$. If N is open in A, then we obtain*

$$\operatorname{Bdry}_A N = A \cap (\operatorname{CL}_X N \setminus N).$$

Proof. We have $\operatorname{Bdry}_A N = \operatorname{CL}_A N \setminus \operatorname{Int}_A N$. Since N is open in A, $\operatorname{Int}_A N = N$. Also, $\operatorname{CL}_A N = A \cap \operatorname{CL}_X N$. Therefore we observe that $\operatorname{Bdry}_A(N) = \operatorname{CL}_A(N) \setminus \operatorname{Int}_A(N) = (A \cap \operatorname{CL}_X N) \setminus N = A \cap (\operatorname{CL}_X N \setminus N)$ as required. □

1.3. **Notation for special spaces.**

Definition 1.4. Let $n \in \mathbb{Z}_{\geq 1}$. Define $\mathbb{D}^n = \{\, x \in \mathbb{R}^n \mid \|x\| \leq 1 \,\}$ and $\mathbb{S}^{n-1} = \{\, x \in \mathbb{R}^n \mid \|x\| = 1 \,\}$ where $\|\cdot\|$ is the norm induced by the standard inner product. Thus $\mathbb{D}^n$ is the closed n-dimensional ball and $\mathbb{S}^{n-1}$ is its boundary, the $(n-1)$-dimensional sphere.

Definition 1.5. Let $n \in \mathbb{Z}_{\geq 1}$ and $i \in \{1, \ldots, n\}$. Define $\mathbb{I}^n = [-1, 1]^n$ and $\mathbb{I}^{n,i}_+ = \{\, x \in \mathbb{I}^n \mid x_i = 1 \,\}$ and $\mathbb{I}^{n,i}_- = \{\, x \in \mathbb{I}^n \mid x_i = -1 \,\}$. We write $\operatorname{Bdry} \mathbb{I}^n$ for the boundary of $\mathbb{I}^n$. Then $\operatorname{Bdry} \mathbb{I}^n = \bigcup_{i=1}^n \mathbb{I}^{n,i}_+ \cup \mathbb{I}^{n,i}_-$ Moreover, $\mathbb{I}^n$ is homeomorphic to $\mathbb{D}^n$, and $\operatorname{Bdry} \mathbb{I}^n$ is homeomorphic to $\mathbb{S}^{n-1}$.

Definition 1.6. Let $n \in \mathbb{Z}_{\geq 1}$ and $k \in \{0, \ldots, n\}$. We denote by $\mathcal{R}(n, k)$ the set of points of $\mathbb{R}^n$ having exactly k rational coordinates. We denote by $\mathcal{M}(n, k)$ the set of points having at most k rational coordinates, and by $\mathcal{L}(n, k)$ the set of points having at least k rational coordinates.

Definition 1.7. Let (X, d) be a metric space, $x \in X$, and $r > 0$. We write $\mathbf{U}(x, r; d)$ for the open ball of radius r centered at x; thus $\mathbf{U}(x, r; d) = \{\, a \in X \mid d(x, a) < r \,\}$. Similarly, we define the closed ball by $\mathbf{B}(x, r; d) = \{\, a \in X \mid d(x, a) \leq r \,\}$.

2. Dimensions

The purpose of this section is to develop the dimension theory that forms the main subject of this manuscript.

2.1. **Dimensions and the decomposition dimension.** We begin by introducing the dimensions considered in this manuscript. Among them, the decomposition dimension is introduced specifically to streamline our proof that the three classical dimensions coincide; it is not a standard notion. Nevertheless, for separable metrizable spaces its value agrees with all three of those dimensions.

Definition 2.1. A subset S of a topological space X is called **clopen** if it is both open and closed. A nonempty topological space X is called **zero-dimensional** if it has a base consisting of clopen sets. Here "zero-dimensional" is used in this sense and does not refer to a particular dimension function. We shall later see that, for each dimension function introduced in this manuscript, having dimension at most zero is equivalent to being zero-dimensional in this sense.

Definition 2.2. Let X be a separable metrizable space. We define $\mathbb{D}\mathrm{im} X = -1$ if and only if $X = \emptyset$. For a nonempty space, write $\mathbb{D}\mathrm{im} X \leq 0$ if X is zero-dimensional. More generally, define $\mathbb{D}\mathrm{im} X$ to be the least integer m for which there exist subspaces $X_0, \dots, X_m$ of X such that

$$\mathbb{D}\mathrm{im} X_i \leq 0 \quad \text{and} \quad X = \bigcup_{i=0}^{m} X_i.$$

Thus $\mathbb{D}\mathrm{im} X \leq n$ means that X is the union of $n+1$ zero-dimensional subspaces. This is a topological invariant, called the **decomposition dimension** of X.

Definition 2.3 (Small inductive dimension)**.** The **small inductive dimension** $\mathrm{ind} X$ of a space X is defined inductively as follows. We set $\mathrm{ind} X = -1$ if and only if $X = \emptyset$. Once the relation $\mathrm{ind} X \leq n-1$ has been defined, we write $\mathrm{ind} X \leq n$ if, for every $x \in X$ and every neighborhood U of x, there exists a neighborhood N of x such that

$$N \subset U \quad \text{and} \quad \mathrm{ind}(\mathrm{Bdry}\, N) \leq n-1.$$

We write $\mathrm{ind} X = n$ if $\mathrm{ind} X \leq n$ but $\mathrm{ind} X \not\leq n-1$.

Definition 2.4 (Large inductive dimension)**.** The **large inductive dimension** $\mathrm{Ind} X$ is defined inductively. We set $\mathrm{Ind} X = -1$ if and only if $X = \emptyset$. Once the relation $\mathrm{Ind} X \leq n-1$ has been defined, we write $\mathrm{Ind} X \leq n$ if, whenever K is closed, U is open, and $K \subset U$, there exists an open set V such that

$$K \subset V \subset \mathrm{CL}\, V \subset U \quad \text{and} \quad \mathrm{Ind}\, \mathrm{Bdry}\, V \leq n-1.$$

We write $\mathrm{Ind} X = n$ if $\mathrm{Ind} X \leq n$ but $\mathrm{Ind} X \not\leq n-1$.

Finally, we introduce the covering dimension. For a family $\mathcal{U}$ of subsets of a set X and a point $x \in X$, let $\deg(\mathcal{U}; x)$ be the number of members of $\mathcal{U}$ containing x, and define $\deg(\mathcal{U}) = \sup_{x \in X} \deg(\mathcal{U}; x)$.

Definition 2.5. Let X be a topological space and $n \in \mathbb{Z}_{\geq 0}$. We write $\dim X \leq n$ if every finite open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ satisfying $\deg(\mathcal{V}) \leq n+1$. We write $n < \dim X$ when $\dim X \leq n$ fails, and write $\dim X = n$ when $\dim X \leq n$ and $n-1 < \dim X$.

We first record some basic properties of the decomposition dimension. The following two facts follow immediately from the definition of $\mathbb{D}\mathrm{im}$.

Lemma 2.1. *Let X be a separable metrizable space and $S \subset X$. Then $\mathbb{D}\mathrm{im} S \leq \mathbb{D}\mathrm{im} X$.*

Proof. If $X = \bigcup_{i=0}^{n} A_i$ with each A_i zero-dimensional, put $B_i = S \cap A_i$. Zero-dimensionality is hereditary. □

Lemma 2.2. *Let X be a separable metrizable space and let $A, B \subset X$. If $\mathbb{D}\mathrm{im} A \leq n$ and $\mathbb{D}\mathrm{im} B \leq 0$, then $\mathbb{D}\mathrm{im}(A \cup B) \leq n+1$.*

Proof. This follows immediately from the definition. □

Proposition 2.3. *Let X be a separable metrizable space. Then $\mathbb{D}\mathrm{im} X \leq 0$ if and only if X has a countable base consisting of clopen sets.*

Proof. This follows from Lemma 1.7. □

Lemma 2.4. *For every separable metrizable space X, the conditions $\mathbb{D}\mathrm{im} X \leq 0$ and $\mathrm{ind} X \leq 0$ are equivalent.*

Lemma 2.5. *Let X be a separable metrizable space. Then $\mathbb{D}\mathrm{im} X \leq 0$ if and only if, for every pair of disjoint closed sets $G, H \subset X$, there exists a clopen set O such that $G \subset O$ and $H \cap O = \emptyset$. In particular, $\mathbb{D}\mathrm{im} X \leq 0$ if and only if $\mathrm{Ind} X \leq 0$.*

Proof. The stated separation property clearly implies $\mathbb{D}\mathrm{im} X \leq 0$, so it remains to prove the converse. Assume $\mathbb{D}\mathrm{im} X \leq 0$. For each $p \in G$, choose a countable neighborhood base $\mathcal{N}_p$ consisting of clopen sets. Since H is closed, we may assume that $V \cap H = \emptyset$ for every $V \in \mathcal{N}_p$, replacing the family by $\{S \cap (X \setminus H)\}_{S \in \mathcal{N}_p}$ if necessary.

Since $\mathbb{D}\mathrm{im}(X \setminus G) \leq 0$, the space $X \setminus G$ has a base $\mathcal{B}$ consisting of sets that are clopen in $X \setminus G$. Because $X \setminus G$ is open in X, every member of $\mathcal{B}$ is also clopen in X. Consider

$$\mathcal{B} \cup \bigcup_{p \in G} \mathcal{N}_p. \tag{2.1}$$

This is a base for X. By Lemma 1.7, it contains a countable base $\{U_i\}_{i \in \mathbb{Z}_{\geq 0}}$. Notice that $U_i \cap G \neq \emptyset$ implies $U_i \cap H = \emptyset$.

Set $V_0 = U_0$, and for $i > 0$ define $V_i = U_i \setminus \bigcup_{j=0}^{i-1} U_j$. The sets V_i are pairwise disjoint and open, and $X = \bigcup_{i \in \mathbb{Z}_{\geq 0}} V_i$. Define

$$O = \bigcup_{V_i \cap G \neq \emptyset} V_i \tag{2.2}$$

The pairwise disjointness of the family gives

$$X \setminus O = \bigcup_{V_i \cap G = \emptyset} V_i.$$

Thus O is clopen and contains G. Moreover, $V_i \cap G \neq \emptyset$ implies $V_i \cap H = \emptyset$, so $H \subset X \setminus O$. Hence O has the required properties. $\square$

Proposition 2.6. *Let X be a nonempty separable metrizable space. Assume that there is a countable family of closed sets $\{A_i\}_{i\in\mathbb{Z}_{\geq 0}}$ such that $\mathbb{D}\mathrm{im} A_i \leq 0$ and $X = \bigcup_{i\in\mathbb{Z}_{\geq 0}} A_i$. Then $\mathbb{D}\mathrm{im} X = 0$.*

Proof. We verify the condition in Lemma 2.5. Let G and H be disjoint closed subsets of X. We inductively construct sets P_i, Q_i, R_i, and L_i for $i \in \mathbb{Z}_{\geq 0}$.

The sets $G \cap A_0$ and $H \cap A_0$ are disjoint closed subsets of A_0. Since $\mathbb{D}\mathrm{im} A_0 \leq 0$, Lemma 2.5 gives clopen subsets P_0, Q_0 of A_0 satisfying

$$G \cap A_0 \subset P_0; \tag{2.3}$$
$$H \cap A_0 \subset Q_0; \tag{2.4}$$
$$P_0 \cap Q_0 = \emptyset; \tag{2.5}$$
$$P_0 \cup Q_0 = A_0. \tag{2.6}$$

Since A_0 is closed, both P_0 and Q_0 are closed in X. Moreover, we see that $G \cup P_0$ and $H \cup Q_0$ are disjoint closed subsets of X. Using the normality of X, choose open sets R_0, L_0 satisfying

$$G \cup P_0 \subset R_0; \tag{2.7}$$
$$H \cup Q_0 \subset L_0; \tag{2.8}$$
$$\mathrm{CL}(R_0) \cap \mathrm{CL}(L_0) = \emptyset. \tag{2.9}$$

In general, construct P_i, Q_i, R_i, and L_i inductively so that

$$P_i \text{ and } Q_i \text{ are clopen subsets of } A_i; \tag{2.10}$$
$$\mathrm{CL}(R_{i-1}) \cap A_i \subset P_i; \tag{2.11}$$
$$\mathrm{CL}(L_{i-1}) \cap A_i \subset Q_i; \tag{2.12}$$
$$P_i \cap Q_i = \emptyset; \tag{2.13}$$
$$P_i \cup Q_i = A_i; \tag{2.14}$$
$$R_i \text{ and } L_i \text{ are open subsets of } X; \tag{2.15}$$
$$\mathrm{CL}(R_{i-1}) \cup P_i \subset R_i; \tag{2.16}$$
$$\mathrm{CL}(L_{i-1}) \cup Q_i \subset L_i; \tag{2.17}$$
$$\mathrm{CL}(R_i) \cap \mathrm{CL}(L_i) = \emptyset. \tag{2.18}$$

Such a construction is possible because each A_i is zero-dimensional and X is normal. For every $i \in \mathbb{Z}_{\geq 0}$ we then have

$$A_i \subset R_i \cup L_i; \tag{2.19}$$

$$\mathrm{CL}(R_{i-1}) \subset R_i; \tag{2.20}$$

$$\mathrm{CL}(L_{i-1}) \subset L_i. \tag{2.21}$$

Set $R = \bigcup_i R_i$ and $L = \bigcup_i L_i$. These are open subsets of X, and $X = \bigcup_i A_i \subset R \cup L$, so $X = R \cup L$. If $x \in R \cap L$, then $x \in R_i \cap L_j$ for some i, j. Taking $k = \max\{i, j\}$ gives $x \in R_k \cap L_k = \emptyset$, a contradiction. Thus $R \cap L = \emptyset$. By construction, $G \subset R$ and $H \subset L$. Since $R \cup L = X$, both R and L are also closed. Lemma 2.5 now implies that $\mathbb{D}\mathrm{im} X = 0$. □

Definition 2.6. Let X be a topological space. A subset S of X is called an F_σ set if it is the union of countably many closed subsets of X.

Corollary 2.7. *If a separable metrizable space is the union of countably many zero-dimensional F_σ subsets, then it is zero-dimensional.*

Theorem 2.8. *Let $n \in \mathbb{Z}_{\geq 0}$ and let X be a separable metrizable space. Assume that X has countably many F_σ subsets $\{A_i\}_{i \in \mathbb{Z}_{\geq 0}}$ such that $\mathbb{D}\mathrm{im} A_i \leq n$ and $X = \bigcup_{i \in \mathbb{Z}_{\geq 0}} A_i$. Then $\mathbb{D}\mathrm{im} X \leq n$.*

Proof. We argue by induction on n. The case $n = 0$ is Proposition 2.6. Assume the theorem for all values less than $n+1$. It suffices to consider a countable family of closed sets $\{A_i\}_{i \in \mathbb{Z}_{\geq 0}}$ such that $\mathbb{D}\mathrm{im} A_i \leq n+1$ and $X = \bigcup_{i \in \mathbb{Z}_{\geq 0}} A_i$.

Set $B_0 = A_0$ and $B_i = A_i \setminus \bigcup_{j=0}^{i-1} A_j$ for $i > 0$. Each B_i is an F_σ set, $\mathbb{D}\mathrm{im} B_i \leq n+1$, the sets B_i are pairwise disjoint, and $X = \bigcup_i B_i$. Write $B_i = M_i \cup N_i$, where $\mathbb{D}\mathrm{im} M_i \leq n$ and $\mathbb{D}\mathrm{im} N_i = 0$.

Put $M = \bigcup_{i \in \mathbb{Z}_{\geq 0}} M_i$ and $N = \bigcup_{i \in \mathbb{Z}_{\geq 0}} N_i$. Since the B_i are pairwise disjoint, we have $M_i = M \cap B_i$, so each M_i is an F_σ subset of M. The inductive hypothesis gives $\mathbb{D}\mathrm{im} M \leq n$. Similarly, we obtain $\mathbb{D}\mathrm{im} N = 0$. Since $X = M \cup N$, Lemma 2.2 yields $\mathbb{D}\mathrm{im} X \leq n+1$. □

We next prove that $\mathbb{D}\mathrm{im} \mathbb{R}^n \leq n$ and $\mathbb{D}\mathrm{im} \mathbb{S}^n \leq n$.

Lemma 2.9. *For every $n \in \mathbb{Z}_{\geq 0}$, the space $\mathbb{Q}^n$ is zero-dimensional.*

Proof. Every singleton is closed and zero-dimensional, so the assertion follows from Proposition 2.6. □

Lemma 2.10. *For every $n \in \mathbb{Z}_{\geq 0}$, the space $(\mathbb{R} \setminus \mathbb{Q})^n$ is zero-dimensional.*

Proof. Fix $x = (x_1, \ldots, x_n) \in (\mathbb{R} \setminus \mathbb{Q})^n$ and $\epsilon > 0$. For each $i \in \{1, \ldots, n\}$, choose $\eta_i, \kappa_i > 0$ satisfying:

(1) $\eta_i < \epsilon$ and $\kappa_i < \epsilon$;
(2) $(x_1 - \eta_1, \ldots, x_n - \eta_n) \in \mathbb{Q}^n$;
(3) $(x_1 + \kappa_1, \ldots, x_n + \kappa_n) \in \mathbb{Q}^n$.

Such numbers exist because $\mathbb{Q}$ is dense in $\mathbb{R}$. Set $U = \prod_{i=1}^{n}(x_i - \eta_i, x_i + \kappa_i)$ and $C = \prod_{i=1}^{n}[x_i - \eta_i, x_i + \kappa_i]$. Then $U \cap (\mathbb{R} \setminus \mathbb{Q})^n = C \cap (\mathbb{R} \setminus \mathbb{Q})^n$ Hence $U \cap (\mathbb{R} \setminus \mathbb{Q})^n$ is a clopen neighborhood of x in $(\mathbb{R} \setminus \mathbb{Q})^n$. Since ϵ was arbitrary, every point has a base of clopen neighborhoods. Thus $\mathbb{D}\mathrm{im}(\mathbb{R} \setminus \mathbb{Q})^n = 0$. □

Lemma 2.11. *Let $n \in \mathbb{Z}_{\geq 1}$ and $k \in \{0, \dots, n\}$. Then we have $\mathbb{D}\mathrm{im}\mathcal{R}(n, k) = 0$.*

Proof. Let $I(n, k)$ be the family of all k-element subsets of $\{1, \dots, n - 1\}$. For $\sigma = \{i_1, \dots, i_k\} \in I(n, k)$ and $v = (v_1, \dots, v_k) \in \mathbb{Q}^k$, let $S(\sigma, v)$ be the set of points of $\mathbb{R}^n$ whose i_j-th coordinate is v_j for every $j \in \{1, \dots, k\}$. Let $T(\sigma, v)$ be the subset of $S(\sigma, v)$ consisting of those points whose remaining coordinates are irrational. Then we have

$$\mathcal{R}(n, k) = \bigcup_{\sigma \in I(n,k)} \bigcup_{v \in \mathbb{Q}^k} T(\sigma, v). \tag{2.22}$$

Since each $S(\sigma, v)$ is closed in $\mathbb{R}^n$ and $S(\sigma, v) \cap \mathcal{R}(n, k) = T(\sigma, v)$, we see that each $T(\sigma, v)$ is closed in $\mathcal{R}(n, k)$. By the fact that $T(\sigma, v)$ is homeomorphic to $(\mathbb{R} \setminus \mathbb{Q})^{n-k}$, Lemma 2.10 shows that $\mathbb{D}\mathrm{im}S(\sigma, v) = 0$. Since $I(n, k)$ and $\mathbb{Q}^k$ are at most countable, Proposition 2.6 gives $\mathbb{D}\mathrm{im}\mathcal{R}(n, k) = 0$. □

Lemma 2.12. *Let $n \in \mathbb{Z}_{\geq 1}$ and $k \in \{0, \dots, n\}$. Then we have $\mathbb{D}\mathrm{im}\mathcal{M}(n, k) \leq k$ and $\mathbb{D}\mathrm{im}\mathcal{L}(n, k) \leq n - k$.*

Proof. We have

$$\mathcal{M}(n, k) = \bigcup_{i=0}^{k} \mathcal{R}(n, i) = \mathcal{R}(n, 0) \cup \mathcal{R}(n, 1) \cup \cdots \cup \mathcal{R}(n, k); \tag{2.23}$$

$$\mathcal{L}(n, k) = \bigcup_{i=k}^{n} \mathcal{R}(n, i) = \mathcal{R}(n, k) \cup \mathcal{R}(n, k + 1) \cup \cdots \cup \mathcal{R}(n, n). \tag{2.24}$$

The conclusion follows from Lemma 2.11. □

Lemma 2.13. *For every $n \in \mathbb{Z}_{\geq 0}$, $\mathbb{D}\mathrm{im}\mathbb{R}^n \leq n$ and $\mathbb{D}\mathrm{im}\mathbb{S}^n \leq n$.*

Proof. Since $\mathbb{R}^n = \bigcup_{k=0}^{n} \mathcal{R}(n, k)$, Lemma 2.11 gives $\mathbb{D}\mathrm{im}\mathbb{R}^n \leq n$ (this also follows from Lemma 2.12). Next consider $\mathbb{S}^n$, the one-point compactification of $\mathbb{R}^n$. We have $\mathbb{S}^n = (\mathcal{R}(n, n) \cup \{\infty\}) \cup \bigcup_{k=0}^{n-1} \mathcal{R}(n, k)$. Proposition 2.6 gives $\mathbb{D}\mathrm{im}(\mathcal{R}(n, n) \cup \{\infty\}) = 0$, and hence we conclude that $\mathbb{D}\mathrm{im}\mathbb{S}^n \leq n$. □

Alternative proof using ind. Let $m \in \mathbb{Z}_{\geq 1}$. Both $\mathbb{R}^{m+1}$ and $\mathbb{S}^{m+1}$ have a base consisting of open sets whose boundaries are homeomorphic to $\mathbb{S}^m$. Hence $\mathbb{D}\mathrm{im}\mathbb{R}^{m+1} \leq \mathbb{D}\mathrm{im}\mathbb{S}^m + 1$ and $\mathbb{D}\mathrm{im}\mathbb{S}^{m+1} \leq \mathbb{D}\mathrm{im}\mathbb{S}^m + 1$. The circle $\mathbb{S}^1$ has a base consisting of open sets whose boundaries contain two points, so $\mathbb{D}\mathrm{im}\mathbb{S}^1 \leq 1$. Alternatively, this follows because $\mathbb{R}$ is the union of the two zero-dimensional sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, and $\mathbb{S}^1$ is the one-point compactification of $\mathbb{R}$. Induction now gives $\mathbb{D}\mathrm{im}\mathbb{R}^n \leq n$ and $\mathbb{D}\mathrm{im}\mathbb{S}^n \leq n$. $\square$

Proposition 2.14. *Every second-countable topological n-dimensional manifold M satisfies $\mathbb{D}\mathrm{im}M \leq n$.*

Proof. The assertion follows from local Euclideanness, second countability, $\mathbb{D}\mathrm{im}\mathbb{R}^n \leq n$, and Theorem 2.8. Alternatively, use a base consisting of open sets whose boundaries are homeomorphic to $\mathbb{S}^{n-1}$ together with Corollary 2.17. $\square$

2.2. **Coincidence of the inductive dimensions and the decomposition dimension.**

Lemma 2.15. *Let X be a separable metrizable space, and $A \subset X$ satisfy $\mathbb{D}\mathrm{im}A = 0$. Assume that K is a closed subset of X, and U is an open subset of X such that $K \subset U$. Then there exists an open set W in X such that $K \subset W \subset U$ and $A \cap \mathrm{Bdry}_X W = \emptyset$.*

Proof. By normality, choose an open set V such that $K \subset V$ and $\mathrm{CL}_X V \subset U$, and put $L = \mathrm{CL}_X V$. Since $\mathbb{D}\mathrm{im}A = 0$, there exists a neighborhood $N \subset A$ of $L \cap A$ such that $L \cap A \subset N \subset U \cap A$ and $\mathrm{Bdry}_A N = \emptyset$.

We show that each of the sets $N \cup K$ and $A \setminus \mathrm{CL}_X(N \cup L)$ is disjoint from the closure of the other. First, clearly

$$\mathrm{CL}_X(N) \cap (A \setminus \mathrm{CL}_X(N \cup L)) = \emptyset. \tag{2.25}$$

Moreover,

$$\begin{aligned}
\mathrm{CL}_X(K) \cap (A \setminus \mathrm{CL}_X(N \cup L)) &= K \cap (A \setminus \mathrm{CL}_X(N \cup L)) \\
&\subset K \cap (A \setminus \mathrm{CL}_X(N)) \\
&\subset K \cap (A \setminus (L \cap A)) \\
&\subset K \cap (A \setminus (K \cap A)) \\
&= K \cap (A \setminus (K \cap A)) \\
&= (K \cap A) \setminus (K \cap A) \\
&= \emptyset.
\end{aligned}$$

Combining these calculations gives $\mathrm{CL}_X(N\cup K)\cap(A\setminus \mathrm{CL}_X(N\cup L)) = \emptyset$. Using $N = N\cap A$, we also obtain

$$\begin{aligned} N\cap \mathrm{CL}_X(A\setminus \mathrm{CL}_X(N\cup L)) &\subset N\cap \mathrm{CL}_X(A\setminus \mathrm{CL}_X(N)) \\ &= N\cap A\cap \mathrm{CL}_X(A\setminus \mathrm{CL}_X(N)) \\ &= N\cap A\cap \mathrm{CL}_A(A\setminus (A\cap \mathrm{CL}_X(N))) \\ &= N\cap A\cap \mathrm{CL}_A(A\setminus \mathrm{CL}_A(N)) \\ &= N\cap \mathrm{CL}_A(A\setminus \mathrm{CL}_A(N)) \\ &= N\cap ((A\setminus \mathrm{CL}_A N)\cup \mathrm{Bdry}_A N) \\ &= N\cap (A\setminus \mathrm{CL}_A N) \\ &= \emptyset. \end{aligned}$$

Since $K\subset V$ and $X\setminus V$ is closed, we have

$$\begin{aligned} K\cap \mathrm{CL}_X(A\setminus \mathrm{CL}_X(N\cup L)) &\subset K\cap \mathrm{CL}_X(A\setminus \mathrm{CL}_X(L)) \\ &= K\cap \mathrm{CL}_X(A\setminus L) \\ &\subset K\cap \mathrm{CL}_X(X\setminus L) \\ &\subset K\cap \mathrm{CL}_X(X\setminus V) \\ &\subset K\cap (X\setminus V) \\ &\subset V\cap (X\setminus V) \\ &= \emptyset. \end{aligned}$$

Thus $K\cap \mathrm{CL}_X(A\setminus \mathrm{CL}_X(N\cup L)) = \emptyset$. It follows that $N\cup K$ and $A\setminus \mathrm{CL}_X(N\cup L)$ are mutually separated. Since X is T_5, there exists an open set W in X such that $N\cup K\subset W$ and $\mathrm{CL}_X W\cap(A\setminus \mathrm{CL}_X(N\cup L)) = \emptyset$. The latter condition implies

$$A\cap \mathrm{CL}_X W\subset A\cap \mathrm{CL}_X(N\cup L). \tag{2.26}$$

Since $\mathrm{CL}_X(N\cup L) = \mathrm{CL}_X N\cup \mathrm{CL}_X L = (\mathrm{CL}_X N)\cup L$ and $L\cap A\subset N$, we obtain

$$A\cap \mathrm{CL}_X W\subset A\cap \mathrm{CL}_X N. \tag{2.27}$$

Replacing W by $V\cap W$ if necessary, we may assume that $W\subset V$. The preceding observations and Lemma 1.9 now give

$$\begin{aligned} A\cap \mathrm{Bdry}_X W = A\cap (\mathrm{CL}_X(W)\cap (X\setminus W)) &\subset A\cap \mathrm{CL}_X(N)\cap (X\setminus N) \\ = A\cap (\mathrm{CL}_X(N)\setminus N) &= \mathrm{Bdry}_A N = \emptyset. \end{aligned}$$

Therefore $A\cap \mathrm{Bdry}_X W = \emptyset$. □

Theorem 2.16. *Let X be a separable metrizable space. Then we have $\mathbb{D}\mathrm{im} X\leq n$ if and only if X has a base $\{V_i\}_{i\in\mathbb{Z}_{\geq 0}}$ such that $\mathbb{D}\mathrm{im}\,\mathrm{Bdry}_X V_i\leq n-1$ for every $i\in\mathbb{Z}_{\geq 0}$.*

Proof. First assume $\mathbb{D}\mathrm{im} X\leq n$. Choose subsets $A_0,\dots,A_n$ of X such that $\mathbb{D}\mathrm{im} A_k\leq 0$ and $\bigcup_{k=0}^{n} A_k = X$. By Lemma 2.15, for every

$k \in \{0, \ldots, n\}$ and every $p \in A_k$ there exists a local base $\mathcal{N}_{p,k}$ of open neighborhoods of p in X such that $A_k \cap \operatorname{Bdry}_X V = \emptyset$ for every $V \in \mathcal{N}_{p,k}$. Hence $\operatorname{Bdry}_X V$ is the union of the sets $A_l \cap \operatorname{Bdry} V$ with $l \neq k$, and therefore $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V \leq n-1$. Consider

$$\bigcup_{k=0}^{n} \bigcup_{p \in A_k} \mathcal{N}_{p,k}. \tag{2.28}$$

This is a base for X. Lemma 1.7 therefore yields a countable base $\{V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ satisfying $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V_i \leq n-1$.

Conversely, assume that X has a base $\{V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ such that

$$\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V_i \leq n-1.$$

By Theorem 2.8, the set $S = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \operatorname{Bdry}_X V_i$ satisfies $\mathbb{D}\mathrm{im} S \leq n-1$. Put $A = X \setminus S$. Then $\{A \cap V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a base for A, and Lemma 1.8 gives $\operatorname{Bdry}_A V_i \subset A \cap \operatorname{Bdry}_X V_i = \emptyset$. Thus $\mathbb{D}\mathrm{im} A \leq 0$. Since $X = S \cup A$, Lemma 2.2 implies $\mathbb{D}\mathrm{im} X \leq n$. □

Theorem 2.16 immediately yields the following.

Corollary 2.17. *For every separable metrizable space X, we have* $\operatorname{ind} X = \mathbb{D}\mathrm{im} X$.

Theorem 2.18. *Let $n \in \mathbb{Z}_{\geq 0}$, and let X be a separable metrizable space. Then the following conditions are equivalent.*

1. $\mathbb{D}\mathrm{im} X \leq n$.
2. *For every closed set K and every open set U in X with $K \subset U$, there exists an open set V such that $K \subset V$, $\mathrm{CL}_X V \subset U$ and* $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V \leq n-1$.
3. *There exists a base $\{V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ for X such that* $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V_i \leq n-1$.

Proof. Proof of [(1) $\Longrightarrow$ (2)]: Assume condition (1), that is, $\mathbb{D}\mathrm{im} X \leq n$. Choose subsets $A_0, \ldots, A_n$ of X such that $\mathbb{D}\mathrm{im} A_i \leq 0$ and $\bigcup_{i=0}^{n} A_i = X$. Let K be closed and U open with $K \subset U$. By Lemma 2.15, there exists an open set V in X such that $K \subset V$, $\mathrm{CL}_X V \subset U$, and $A_0 \cap \operatorname{Bdry}_X V = \emptyset$. The definition of $\mathbb{D}\mathrm{im}$ then gives $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V \leq n-1$, proving (2).

Proof of [(2) $\Longrightarrow$ (3)]: Choose a countable base $\mathcal{B} = \{B_i\}_{i \in \mathbb{Z}_{\geq 0}}$ for X, and define

$$\mathcal{C} = \{\, (B_i, B_k) \mid B_i, B_k \in \mathcal{B}, \mathrm{CL}(B_i) \subset B_k \,\}. \tag{2.29}$$

The set $\mathcal{C}$ is at most countable. For every $(B_i, B_k) \in \mathcal{C}$, apply (2) to $\mathrm{CL}_X B_i$ and B_k to obtain an open set $V_{i,k}$ such that $\mathrm{CL}_X B_i \subset V_{i,k}$, $\mathrm{CL}\, V_{i,k} \subset B_k$, and $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V_{i,k} \leq n-1$. The family $\{V_{i,k}\}$ is a base for X, which proves (3).

Proof of [(3) $\Longrightarrow$ (1)]: Assume that X has a base $\{V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ satisfying $\mathbb{D}\mathrm{im}\operatorname{Bdry}_X V_i \leq n-1$. Theorem 2.8 implies that $S = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \operatorname{Bdry}_X V_i$

satisfies $\mathbb{D}\mathrm{im} S \leq n-1$. Put $A = X \setminus S$. Then $\{A \cap V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a base for A, and Lemma 1.8 gives $\mathrm{Bdry}_A V_i \subset A \cap \mathrm{Bdry}_X V_i = \emptyset$. Hence $\mathbb{D}\mathrm{im} A \leq 0$. Since $X = S \cup A$, Lemma 2.2 yields $\mathbb{D}\mathrm{im} X \leq n$. □

Theorem 2.18 yields the following.

Corollary 2.19. *For every separable metrizable space X, we have* $\mathrm{ind} X = \mathrm{Ind} X = \mathbb{D}\mathrm{im} X$.

We conclude our discussion of inductive dimensions by proving an inequality between the inductive and covering dimensions. We begin with a lemma that lifts open subsets of a subspace to the ambient space.

Lemma 2.20 ([7])**.** *Let (X, d) be a metric space, A be a subspace of X, and let U be open in A. Define*

$$\mathfrak{B}(U) = \{\, x \in X \mid d(x, U) < d(x, A \setminus U) \,\}.$$

Then $\mathfrak{B}(U)$ is open in X and $\mathfrak{B}(U) \cap A = U$. Moreover, if U and V are disjoint open subsets of A, then $\mathfrak{B}(U) \cap \mathfrak{B}(V) = \emptyset$.

Proof. It is immediate that $\mathfrak{B}(U)$ is open. We show that $\mathfrak{B}(U) \cap A = U$. If $x \in \mathfrak{B}(U) \cap A$, then $0 < d(x, A \setminus U)$, so $x \notin A \setminus U$. Since $x \in A$, it follows that $x \in U$. Conversely, if $x \in U$, then $d(x, U) = 0$ and, because U is open in A, $d(x, A \setminus U) > 0$. Hence $x \in \mathfrak{B}(U) \cap A$.

Now let U and V be disjoint open subsets of A. Since $V \subset A \setminus U$ and $U \subset A \setminus V$, for every $x \in X$ we have $d(x, A \setminus U) \leq d(x, V)$ and $d(x, A \setminus V) \leq d(x, U)$. Therefore $\mathfrak{B}(U) \cap \mathfrak{B}(V) = \emptyset$. □

Theorem 2.21. *Let X be a separable metrizable space with $\mathbb{D}\mathrm{im} X \leq n$, and let $\mathcal{U}$ be an open cover of X. Then there exist a locally finite open cover $\mathcal{V}$ of X refining $\mathcal{U}$ and $n+1$ families of open sets (not necessarily covers) $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n$ such that each $\mathcal{W}_i$ is pairwise disjoint and*

$$\mathcal{V} = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_n.$$

Proof. Fix a metric d inducing the topology of X. Since X is paracompact, there exists a locally finite open cover $\mathcal{C}$ refining $\mathcal{U}$. Since X is separable and metrizable, it is Lindelöf, so we may assume that $\mathcal{C}$ is countable. Write $\mathcal{C} = \{U_k\}_{k \in \mathbb{Z}_{\geq 0}}$.

Since $\mathbb{D}\mathrm{im} X \leq n$, there exist $(n+1)$-many zero-dimensional subspaces $\{A_i\}_{i=0}^{n}$ such that $X = \bigcup_{i=0}^{n} A_i$. For each $i \in \mathbb{Z}_{\geq 0}$, consider the cover $\{A_i \cap U_k\}_{k \in \mathbb{Z}_{\geq 0}}$ of A_i. Since A_i is zero-dimensional and separable, this cover admits an at most countable clopen refinement $\{O_l^{(i)}\}_{l \in \mathbb{Z}_{\geq 0}}$ covering A_i. Define

$$G_0^{(i)} = O_0^{(i)}, \ G_j^{(i)} = O_j^{(i)} \setminus \bigcup_{l<j} O_l^{(i)} \ (j \geq 2). \tag{2.30}$$

The family $\{G_j^{(i)}\}_{j \in \mathbb{Z}_{\geq 0}}$ consists of sets open in A_i (they are also closed in A_i, although we do not use this), covers A_i, and refines $\{A_i \cap$

$U_k\}_{k\in\mathbb{Z}_{\geq 0}}$. It is pairwise disjoint by construction. We next define $\{I_a^{(i)}\}_{a\in\mathbb{Z}_{\geq 0}}$ inductively. For $a=0$, set

$$I_0^{(i)} = \{\, j\in\mathbb{Z}_{\geq 0} \mid G_j \subset U_1 \,\}. \tag{2.31}$$

Once $I_0^{(i)},\ldots,I_{m-1}^{(i)}$ have been defined, set

$$I_m^{(i)} = \{\, j\in\mathbb{Z}_{\geq 0} \mid G_j \subset U_m \,\} \setminus \bigcup_{a=1}^{m-1} I_a^{(i)}. \tag{2.32}$$

For $a\in\mathbb{Z}_{\geq 0}$, define

$$P_a^{(i)} = \bigcup_{s\in I_a} G_s^{(i)}. \tag{2.33}$$

Each $P_a^{(i)}$ is open in A_i, and $\{P_a^{(i)}\}_{a\in\mathbb{Z}_{\geq 0}}$ is a pairwise disjoint cover of A_i. Define $Q_m^{(i)} = \mathfrak{B}(P_m^{(i)})\cap U_m$; see Lemma 2.20. Lemma 2.20 implies that $\{Q_m^{(i)}\}_{m\in\mathbb{Z}_{\geq 0}}$ is pairwise disjoint and covers A_i, although some $Q_m^{(i)}$ may be empty. Since $Q_m^{(i)} \subset U_m$, this family refines $\{U_i\}_{i\in\mathbb{Z}_{\geq 0}}$. For $x\in X$, let N be a neighborhood witnessing the local finiteness of $\mathcal{C}$. Then

$$N\cap Q_m^{(i)} \neq \emptyset \Longrightarrow N\cap U_m \neq \emptyset. \tag{2.34}$$

It follows that $\{Q_m^{(i)}\}_{m\in\mathbb{Z}_{\geq 0}}$ is locally finite, although it need not cover X. For each $i\in\{0,1,\ldots,n\}$, define $\mathcal{W}_i = \{Q_m^{(i)}\}_{m\in\mathbb{Z}_{\geq 0}}$. Then $\mathcal{W}_i$ is pairwise disjoint, locally finite, and refines $\mathcal{C}=\{U_k\}_{k\in\mathbb{Z}_{\geq 0}}$.

Put $\mathcal{V} = \mathcal{W}_0\cup\mathcal{W}_1\cup\cdots\cup\mathcal{W}_n$. Since this is a finite union of locally finite families refining $\mathcal{C}$, it is locally finite and refines $\mathcal{C}$. Moreover, $A_i \subset \bigcup\mathcal{W}_i$ and $X=\bigcup_{i=0}^n A_i$, so $\mathcal{V}$ covers X. Finally, the family $\mathcal{C}$ refines $\mathcal{U}$, and therefore so does $\mathcal{V}$. This completes the proof. □

Theorem 2.22. *For every separable metrizable space X, we have the inequality* $\mathrm{dim}X \leq \mathrm{ind}X$*.*

Proof. This follows from Theorem 2.21. □

Later in this manuscript we shall prove that equality holds.

2.3. **Covering dimension in normal spaces.** In this subsection, we first discuss covering dimension for normal spaces in general and then specialize to separable metrizable spaces.

Lemma 2.23. *Let X be a normal space. Then the following conditions are equivalent.*

(1) $\mathrm{dim}X \leq n$
(2) *For every finite open cover $\{U_1,\ldots,U_m\}$ of X, there exists an open cover $\mathcal{V}=\{V_1,\ldots,V_m\}$ such that $V_i\subset U_i$ and* $\deg(\mathcal{V}) \leq n+1$*.*

(3) *For every finite open cover* $\{U_1, \dots, U_m\}$ *of* X*, there exists an open cover* $\mathcal{V} = \{V_1, \dots, V_m\}$ *such that* $V_i \subset U_i$ *and* $\bigcap_{j \in J} V_j = \emptyset$ *for every subset* $J \subset \{1, \dots, m\}$ *of cardinality* $n+2$*.*

(4) *For every open cover* $\{U_1, \dots, U_{n+2}\}$ *of* X *of cardinality* $n+2$*, there exists an open cover* $\mathcal{V} = \{V_1, \dots, V_{n+2}\}$ *such that* $V_i \subset U_i$ *and* $\bigcap_{j=1}^{n+2} V_j = \emptyset$*.*

Proof. The equivalence $[(2) \iff (3)]$ is immediate. The implications $[(2) \implies (1)]$ and $[(3) \implies (4)]$ are also immediate.

We prove $[(1) \implies (2)]$. Given an open cover $\{U_1, \dots, U_m\}$, choose an open refinement $\mathcal{G} = \{G_1, \dots, G_k\}$ such that $\deg(\mathcal{G}) \leq n+1$. Define $l\colon \{1, \dots, k\} \to \{1, \dots, m\}$ by letting $l(a)$ be the least b such that $G_a \subset U_b$. For each $i \in \{1, \dots, m\}$, put $V_i = \bigcup_{l(a)=i} G_a$ and $\mathcal{V} = \{V_1, \dots, V_m\}$. Then $\mathcal{V}$ is an open cover of X, and $\deg(\mathcal{G}) \leq n+1$ implies $\deg(\mathcal{V}) \leq n+1$.

We next prove $[(4) \implies (3)]$. Let $\{U_1, \dots, U_m\}$ be an arbitrary finite open cover of X. If $m < n+2$, we may add empty sets and assume that $n+2 \leq m$. Let $\mathcal{A}$ be the collection of all partitions of $\{1, \dots, m\}$ into $(n+2)$-many nonempty subsets. Thus, each member of $\mathcal{A}$ is a family $\{I_1, \dots, I_{n+2}\}$ such that every I_k is a nonempty subset of $\{1, \dots, m\}$, $I_a \cap I_b = \emptyset$ whenever $a \neq b$, and $\bigcup_{k=1}^{n+2} I_k = \{1, \dots, m\}$. Since $\mathcal{A}$ is finite, write $\mathcal{A} = \{A_1, \dots, A_r\}$.

Inductively we construct open covers $\{G_{l,1}, \dots, G_{l,m}\}$ of X for $l \in \{0, 1, \dots, r\}$. First, define

$$G_{0,i} = U_i. \tag{2.35}$$

Assume that $\{G_{l,1}, \dots, G_{l,m}\}$ has been constructed, and write $A_l = \{I_1, \dots, I_{n+2}\}$. For $k \in \{1, \dots, n+2\}$, put $O_k = \bigcup_{j \in I_k} G_{l,j}$. Then $\{O_1, \dots, O_{n+2}\}$ is an open cover of X. By (4), it has an open refinement $\{P_1, \dots, P_{n+2}\}$ such that $\bigcap_{i=1}^{n+2} P_i = \emptyset$. For $i \in \{1, \dots, m\}$, define

$$G_{l+1,i} = P_{k(i)} \cap G_{l,i}. \tag{2.36}$$

Here $k(i) \in \{1, \dots, n+2\}$ is determined by $i \in I_{k(i)}$. Notice that $G_{l+1,i} \subset G_{l,i}$.

Next, we put $V_i = G_{r,i}$ and $\mathcal{V} = \{V_1, \dots, V_m\}$. We show that $\deg(\mathcal{V}) \leq n+1$. Let $K = \{a_1, \dots, a_{n+2}\}$ be any subset of $\{1, \dots, m\}$ of cardinality $n+2$. Set $I_1 = \{a_1\}, \dots, I_{n+1} = \{a_{n+1}\}$, and let I_{n+2} consist of all remaining elements of $\{1, \dots, m\}$. Take l such that $A_l = \{I_1, \dots, I_{n+2}\}$, and use the notation from the construction above. For every $a_i \in K$, we have

$$V_{a_i} = G_{r,a_i} \subset G_{l+1,a_i} \subset G_{l,a_i} \cap P_i. \tag{2.37}$$

Since $\bigcap_{i=1}^{n+2} P_i = \emptyset$, it follows that $\bigcap_{a \in K} V_a = \emptyset$. Thus (3) holds. $\square$

Lemma 2.24. *Let* X *be a normal space. Then the following conditions are equivalent.*

(1) $\dim X \le n$
(2) *For every finite open cover* $\{U_1, \ldots, U_m\}$ *of* X*, there exists an open cover* $\mathcal{V} = \{V_1, \ldots, V_m\}$ *such that* $\mathrm{CL}\, V_i \subset U_i$ *and* $\deg(\mathcal{V}) \le n+1$.
(3) *For every open cover* $\{U_1, \ldots, U_{n+2}\}$ *of* X *of cardinality* $n+2$*, there exists a closed cover* $\mathcal{V} = \{F_1, \ldots, F_{n+2}\}$ *such that* $F_i \subset U_i$ *and* $\bigcap_{j=1}^{n+2} F_j = \emptyset$.

Proof. [(1) $\iff$ (2)] follows from Lemma 1.5. The implication [(2) $\implies$ (3)] is immediate, and [(3) $\implies$ (1)] follows from Lemma 1.6. □

Covering dimension can also be characterized by the degree of the boundaries of a family.

Proposition 2.25. *Let* X *be a normal space. Then the following conditions are equivalent. Notice that none of the families appearing below is assumed to be a cover.*

(1) $\dim X \le n$
(2) *For every finite family* $\{U_1, \ldots, U_m\}$ *of open subsets of* X *and every family* $\{F_1, \ldots, F_m\}$ *of closed subsets with* $F_i \subset U_i$*, there exists a family* $\mathcal{V} = \{V_1, \ldots, V_m\}$ *of open sets such that* $F_i \subset V_i$*,* $\mathrm{CL}\, V_i \subset U_i$*, and* $\deg(\{\mathrm{Bdry}\, V_1, \ldots, \mathrm{Bdry}\, V_m\}) \le n$.
(3) *For every finite family* $\{U_1, \ldots, U_m\}$ *of open subsets of* X *and every family* $\{F_1, \ldots, F_m\}$ *of closed subsets with* $F_i \subset U_i$*, there exist families* $\mathcal{W} = \{W_1, \ldots, W_m\}$ *and* $\mathcal{V} = \{V_1, \ldots, V_m\}$ *of open sets such that* $F_i \subset V_i \subset \mathrm{CL}\, V_i \subset W_i$*,* $\mathrm{CL}\, W_i \subset U_i$*, and*
$$\deg(\{\mathrm{CL}\, W_1 \setminus V_1, \ldots, \mathrm{CL}\, W_m \setminus V_m\}) \le n.$$
(4) *For every family* $\{U_1, \ldots, U_{n+1}\}$ *of* $n+1$ *open subsets of* X *and every family* $\{F_1, \ldots, F_{n+1}\}$ *of closed subsets with* $F_i \subset U_i$*, there exists a family* $\mathcal{V} = \{V_1, \ldots, V_{n+1}\}$ *of open sets such that* $F_i \subset V_i$*,* $\mathrm{CL}\, V_i \subset U_i$*, and* $\bigcap_{j=1}^{n+1} \mathrm{Bdry}\, V_j = \emptyset$.

Proof. Proof of [(1) $\implies$ (2)]: Let $\mathcal{H}$ be the family of all sets that can be expressed as the intersection of an m-element subfamily of
$$\{U_1, \ldots, U_m\} \cup \{X \setminus F_1, \ldots, X \setminus F_m\}.$$
By the assumption, $\mathcal{H}$ is an open cover of X. Hence (1) gives an open refinement $\{W_1, \ldots, W_q\}$ of $\mathcal{H}$ whose degree is at most $n+1$. Choose a closed cover $\{K_1, \ldots, K_q\}$ such that $K_r \subset W_r$. For $r \in \{1, \ldots, q\}$, let N_r be the set of all $i \in \{1, \ldots, m\}$ such that $F_i \cap W_r \ne \emptyset$. By Lemma 1.5, for each $i \in N_r$ choose $V_{i,r}$ so that $K_r \subset V_{i,r} \subset \mathrm{CL}\, V_{i,r} \subset W_r$. We may choose them so that $\mathrm{CL}\, V_{i,r} \subset V_{j,r}$ whenever $i, j \in N_r$ and $i < j$. For $i \in \{1, \ldots, m\}$, define
$$V_i = \bigcup \{V_{i,r} \mid i \in N_r\}. \tag{2.38}$$

Then $F_i \subset V_i$: indeed, if $x \in F_i$, choose r with $x \in K_r$; then $x \in V_{i,r} \subset V_i$. Moreover, the definitions of N_r and $\mathcal{H}$ show that $W_r \subset U_i$ whenever $i \in N_r$. Since the union defining V_i is finite, $\mathrm{CL}\, V_i \subset U_i$.

Suppose, toward a contradiction, that there exists a subset $I \subset \{1, \dots, m\}$ of cardinality $n+1$, say $I = \{u_1, \dots, u_{n+1}\}$, such that

$$\bigcap_{i=1}^{n+1} \mathrm{Bdry}\, V_{u_i} \neq \emptyset. \tag{2.39}$$

Choose $x \in \bigcap_{i=1}^{n+1} \mathrm{Bdry}\, V_{u_i}$. For each $i \in \{1, \dots, n+1\}$, choose $r(i)$ such that $x \in \mathrm{Bdry}\, V_{u_i, r(i)}$. If $i \neq j$, then $r(i) \neq r(j)$. Indeed, supposing $i < j$ and $r(i) = r(j)$ would give $\mathrm{CL}\, V_{u_i, r(i)} \subset V_{u_j, r(i)}$, contradicting $x \in \mathrm{Bdry}\, V_{u_j}$.

Since $x \in \mathrm{Bdry}\, V_{u_i, r(i)}$, in particular $x \notin V_{u_i, r(i)}$. Choose $r(n+2)$ such that $x \in K_{r(n+2)}$, which is possible because $\{K_r\}_{r=1}^{q}$ covers X. Then $r(n+2)$ is distinct from all the $r(i)$. Since $\mathrm{Bdry}\, V_{u_i} \subset W_{r(i)}$, this yields $x \in \bigcap_{i=1}^{n+2} W_{r(i)}$, contradicting $\deg(\{W_1, \dots, W_q\}) \leq n+1$. Therefore we conclude $\deg(\{\mathrm{Bdry}\, V_1, \dots, \mathrm{Bdry}\, V_m\}) \leq n$.

Proof of $[(2) \implies (3)]$: By (2), there exists a family $\{V_1, \dots, V_m\}$ of open sets satisfying $F_i \subset V_i$, $\mathrm{CL}\, V_i \subset U_i$, and

$$\deg(\{\mathrm{Bdry}\, V_1, \dots, \mathrm{Bdry}\, V_m\}) \leq n. \tag{2.40}$$

Since $\mathrm{Bdry}\, V_i \subset U_i$, Lemma 1.6 gives a family $\{P_1, \dots, P_m\}$ of open sets such that $\{\mathrm{CL}\, P_1, \dots, \mathrm{CL}\, P_m\}$ is similar to $\{\mathrm{Bdry}\, V_1, \dots, \mathrm{Bdry}\, V_m\}$ and $\mathrm{Bdry}\, V_i \subset P_i \subset U_i$. Put $W_i = V_i \cup P_i$. Then $\mathrm{CL}\, V_i \subset W_i$, $\mathrm{Bdry}\, V_i \subset \mathrm{CL}\, W_i \backslash V_i$, and $\mathrm{CL}\, W_i \backslash V_i \subset \mathrm{CL}\, P_i$. Hence $\{\mathrm{CL}\, W_1 \backslash V_1, \dots, \mathrm{CL}\, W_m \backslash V_m\}$ is similar to $\{\mathrm{Bdry}\, V_1, \dots, \mathrm{Bdry}\, V_m\}$, and therefore

$$\deg(\{\mathrm{CL}\, W_1 \setminus V_1, \dots, \mathrm{CL}\, W_m \setminus V_m\}) \leq n. \tag{2.41}$$

The implication $[(3) \implies (4)]$ is immediate.

Proof of $[(4) \implies (1)]$: Let $\{O_1, \dots, O_{n+2}\}$ be an arbitrary open cover of X. By Lemma 1.5, there exists a closed cover $\{A_1, \dots, A_{n+2}\}$ such that $A_i \subset O_i$. Apply (4) to $\{O_1, \dots, O_{n+1}\}$ and $\{A_1, \dots, A_{n+1}\}$ to obtain a family $\{H_1, \dots, H_{n+1}\}$ of open sets such that $A_i \subset H_i$, $\mathrm{CL}\, H_i \subset O_i$, and $\bigcap_{i=1}^{n+1} \mathrm{Bdry}\, H_i = \emptyset$. For $i \in \{1, \dots, n+2\}$, define a closed set L_i by $L_i = \mathrm{CL}\, H_i \setminus \bigcup_{k<i} H_k$ when $i = 1, \dots, n+1$, and $L_{n+2} = A_{n+2} \setminus \bigcup_{k=1}^{n+1} H_k$. Then $\{L_1, \dots, L_{n+2}\}$ is a closed cover of X. Indeed, if x belongs to no H_i, then $x \in L_{n+2}$; otherwise, if j is the least index such that $x \in H_j$, then $x \in L_j$. Moreover, $L_i \subset O_i$. Finally,

$$\bigcap_{i=1}^{n+2} L_i = L_{n+2} \cap \bigcap_{i=1}^{n+1} \left(\mathrm{CL}\, H_i \setminus \bigcup_{k<i} H_k \right) \subset L_{n+2} \cap \bigcap_{i=1}^{n+1} \mathrm{Bdry}\, H_i = \emptyset.$$

Thus Lemma 2.24 gives $\dim X \leq n$. $\square$

Definition 2.7. Let X be a topological space, and let A, B, and C be its subsets. Assume that $A \cap B = \emptyset$. We say that C **separates**

A **and** B if there exist open sets U and V such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$, and $X \setminus C = U \coprod V$. Notice that C is necessarily closed.

Proposition 2.26. *(Eilenberg–Otto [3]) Let X be a normal space. Then the following conditions are equivalent.*

(1) $\dim X \leq n$
(2) *For every $(n+1)$-many pairs $(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})$ of disjoint closed sets, there exist closed sets $C_1, \ldots, C_{n+1}$ such that each C_i separates A_i and B_i and $\bigcap_{i=1}^{n+1} C_i = \emptyset$.*

Proof. Proof of $[(1) \implies (2)]$: We first apply Proposition 2.25 to the family $\{A_1, \ldots, A_{n+1}\}$ of closed sets and the family $\{X \setminus B_1, \ldots, X \setminus B_{n+1}\}$ of open sets. We obtain open sets $\{V_1, \ldots, V_{n+1}\}$ such that $A_i \subset V_i$, $\mathrm{CL}\, V_i \subset X \setminus B_i$, and $\bigcap_{i=1}^{n+1} \mathrm{Bdry}\, V_i = \emptyset$. Taking $C_i = \mathrm{Bdry}\, V_i$ proves (2).

Proof of $[(2) \implies (1)]$: Let $\{F_1, \ldots, F_{n+1}\}$ be a family of closed sets and $\{U_1, \ldots, U_{n+1}\}$ a family of open sets such that $F_i \subset U_i$. Apply (2) to the $n+1$ pairs $(F_1, X \setminus U_1), \ldots, (F_{n+1}, X \setminus U_{n+1})$. This gives closed sets C_i separating F_i and $X \setminus U_i$ such that $\bigcap_{i=1}^{n+1} C_i = \emptyset$. Choose open sets V_i and V_i' such that $V_i \sqcup C_i \sqcup V_i' = X$, $F_i \subset V_i$, and $X \setminus U_i \subset V_i'$. Then $\mathrm{Bdry}\, V_i \subset C_i$ and $\mathrm{CL}\, V_i \subset U_i$, and hence

$$\bigcap_{i=1}^{n+1} \mathrm{Bdry}\, V_i \subset \bigcap_{i=1}^{n+1} C_i = \emptyset.$$

Proposition 2.25 now gives $\dim X \leq n$. $\square$

Theorem 2.27. *Let $n \in \mathbb{Z}_{\geq 0}$, let X be a normal space with $\dim X \leq n$, and let $\{F_i\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\{U_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be families of closed and open sets, respectively, such that $F_i \subset U_i$. Then there exists a family $\{V_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of open sets such that $F_i \subset V_i$, $\mathrm{CL}\, V_i \subset U_i$, and $\deg(\{\mathrm{Bdry}\, V_i\}_{i \in \mathbb{Z}_{\geq 0}}) \leq n$.*

Proof. Consider all subsets of $\mathbb{Z}_{\geq 0}$ of cardinality $n+1$. There are at most countably many such subsets, so enumerate them as

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_i, \ldots \tag{2.42}$$

We inductively construct sequences of families of open sets $\{P_i(k)\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\{Q_i(k)\}_{i \in \mathbb{Z}_{\geq 0}}$, indexed by $k \in \mathbb{Z}_{\geq 0}$, such that for every $i \in \mathbb{Z}_{\geq 0}$, $F_i \subset P_i(k)$ and $\mathrm{CL}\, Q_i(k) \subset U_i$, and such that for every $l \in \{1, \ldots, k\}$,

$$\deg(\{\mathrm{CL}(Q_a(k)) \setminus P_a(k) \mid a \in \Gamma_l\}) \leq n. \tag{2.43}$$

Assume that these families have been constructed through stage k. For $a \in \Gamma_{k+1}$, apply Proposition 2.25 to $\{\mathrm{CL}\, P_a(k)\}_{a \in \Gamma_{k+1}}$ and $\{Q_a(k)\}_{a \in \Gamma_{k+1}}$. Choose $\{P_a(k+1)\}_{a \in \Gamma_{k+1}}$ and $\{Q_a(k+1)\}_{a \in \Gamma_{k+1}}$ so that

$$\mathrm{CL}\, P_a(k) \subset P_a(k+1) \subset Q_a(k+1), \qquad \mathrm{CL}\, Q_a(k+1) \subset Q_a(k),$$

and

$$\deg(\{\mathrm{CL}\, Q_a(k+1) \setminus P_a(k+1)\}_{a \in \Gamma_{k+1}}) \leq n. \tag{2.44}$$

For $b \in \mathbb{Z}_{\geq 0} \setminus \Gamma_{k+1}$, define $P_b(k+1) = P_b(k)$ and $Q_b(k+1) = Q_b(k)$. Since

$$\mathrm{CL}\, Q_i(k+1) \setminus P_i(k+1) \subset \mathrm{CL}\, Q_i(k) \setminus P_i(k)$$

for every $i \in \mathbb{Z}_{\geq 0}$, the following holds for every l:

$$\deg(\{\mathrm{CL}(Q_a(k+1)) \setminus P_a(k+1) \mid a \in \Gamma_l\}) \leq n. \tag{2.45}$$

Next, for each $i \in \mathbb{Z}_{\geq 0}$, now define $V_i = \bigcup_{k \in \mathbb{Z}_{\geq 0}} P_i(k)$. Clearly, $F_i \subset V_i \subset \mathrm{CL}\, V_i \subset U_i$. Moreover, for every $k \in \mathbb{Z}_{\geq 0}$, we have $P_i(k) \subset V_i$ and $\mathrm{CL}\, V_i \subset Q_i(k)$, and hence $\mathrm{Bdry}\, V_i \subset \mathrm{CL}(Q_i(k)) \setminus P_i(k)$. Therefore, for every Γ_k, we obtain

$$\deg(\{\mathrm{Bdry}\, V_a \mid a \in \Gamma_k\}) \leq n. \tag{2.46}$$

This completes the proof. □

2.4. **Separable metrizable spaces and covering dimension.**

Theorem 2.28. *Let X be a separable metrizable space, and let $n \in \mathbb{Z}_{\geq 0}$. Then the following conditions are equivalent.*

(1) $\dim X \leq n$.
(2) *The space X has a base $\mathcal{B}$ satisfying* $\deg(\{\, \mathrm{Bdry}\, B \mid B \in \mathcal{B}\,\}) \leq n$.
(3) $\mathrm{ind} X \leq n$.

In particular, we have $\mathrm{ind} X \leq \dim X$.

Proof. Proof of [(1) $\Longrightarrow$ (2)]: Choose countable families $\{F_s\}_{s \in \mathbb{Z}_{\geq 0}}$ of closed subsets of X and $\{U_s\}_{s \in \mathbb{Z}_{\geq 0}}$ of open subsets of X such that both $\{\mathrm{Int}\, F_s\}_{s \in \mathbb{Z}_{\geq 0}}$ and $\{U_s\}_{s \in \mathbb{Z}_{\geq 0}}$ are bases for X, and such that $F_s \subset U_s$ for every $s \in \mathbb{Z}_{\geq 0}$. Such families exist because X is separable and metrizable.

Apply Theorem 2.27 to the two countable families $\{F_s\}_{s \in \mathbb{Z}_{\geq 0}}$ and $\{U_s\}_{s \in \mathbb{Z}_{\geq 0}}$. We obtain a family $\{V_s\}_{s \in \mathbb{Z}_{\geq 0}}$ of open sets such that $F_s \subset V_s$ and $\mathrm{CL}\, V_s \subset U_s$ for every $s \in \mathbb{Z}_{\geq 0}$, and

$$\deg(\{\mathrm{Bdry}\, V_s\}_{s \in \mathbb{Z}_{\geq 0}}) \leq n. \tag{2.47}$$

Since $F_s \subset V_s$ and $\mathrm{CL}\, V_s \subset U_s$ for every $s \in \mathbb{Z}_{\geq 0}$, while both $\{\mathrm{Int}\, F_s\}_{s \in \mathbb{Z}_{\geq 0}}$ and $\{U_s\}_{s \in \mathbb{Z}_{\geq 0}}$ are bases for X, the family $\{V_s\}_{s \in \mathbb{Z}_{\geq 0}}$ is also a base for X. Thus (2) holds.

We next prove [(2) $\Longrightarrow$ (3)] by induction. When $n = 0$, the implication follows from the definition of ind; that is, [(2) $\Longrightarrow$ (3)] holds. Assume that [(2) $\Longrightarrow$ (3)] has been proved through $n-1$, and consider the case of n. By (2), choose a base $\mathcal{B}$ for X satisfying

$$\deg(\{\, \mathrm{Bdry}\, B \mid B \in \mathcal{B}\,\}) \leq n \tag{2.48}$$

Fix $A \in \mathcal{B}$. Then $\{ B \cap \operatorname{Bdry} A \mid B \in \mathcal{B} \}$ is a base for $\operatorname{Bdry} A$. By Lemma 1.8,

$$\operatorname{Bdry}_{\operatorname{Bdry} A}(B \cap \operatorname{Bdry} A) \subset \operatorname{Bdry}(B) \cap \operatorname{Bdry} A. \tag{2.49}$$

Consequently, if we choose n distinct elements $B_1, \ldots, B_n \in \mathcal{B} \setminus \{A\}$ then (2.48) gives

$$\bigcap_{i=1}^{n} \operatorname{Bdry}_{\operatorname{Bdry} A}(B_i \cap \operatorname{Bdry} A) \subset \operatorname{Bdry} A \cap \bigcap_{i=1}^{n} \operatorname{Bdry} B_i = \emptyset. \tag{2.50}$$

Thus $S = \operatorname{Bdry} A$ has a base $\mathcal{C} = \{ B \cap S \mid B \in \mathcal{B} \}$ satisfying $\deg(\{ \operatorname{Bdry}_S B \mid B \in \mathcal{C} \}) \leq n-1$. The induction hypothesis for the case of $n-1$ gives $\operatorname{ind} \operatorname{Bdry} A \leq n-1$. Since $A \in \mathcal{B}$ was arbitrary and $\mathcal{B}$ is a base for X, the definition of ind gives $\operatorname{ind} X \leq n$. This proves (3) for n.

Finally, the implication [(3) $\Longrightarrow$ (1)] was already proved in Theorem 2.22. $\square$

Theorem 2.29. *Let X be a separable metrizable space. Then*

$$\operatorname{ind} X = \operatorname{Ind} X = \dim X = \mathbb{D}\mathrm{im} X. \tag{2.51}$$

Proof. This follows from Corollary 2.19, Theorem 2.22, and Theorem 2.28. $\square$

2.5. **Covering dimension characterized by extension to spheres under countable paracompactness.** A topological space X is called **countably paracompact** if $X \times [0,1]$ is normal. There is also a characterization in terms of covers, but in this subsection we use only the normality of $X \times [0,1]$. Every metrizable space is countably paracompact.

Theorem 2.30 (Borsuk's homotopy extension theorem)**.** *Let $n \in \mathbb{Z}_{\geq 0}$, X be a countably paracompact normal space, and A be a closed subset of X. For a continuous map $H\colon A \times [0,1] \to \mathbb{S}^n$, put $H(x,0) = f(x)$ and $H(x,1) = g(x)$. If there exists a map $F\colon X \to \mathbb{S}^n$ such that $F|_A = f$, then there exists a map $G\colon X \to \mathbb{S}^n$ such that $G|_A = g$. Moreover, the map G can be chosen to be homotopic to F. Thus, extendability to the whole space is invariant under homotopies taking values in the sphere.*

Proof. Since X is countably paracompact, the product $X \times [0,1]$ is normal. Using the map F from the hypothesis, extend the domain of the map H given in the statement: $H\colon A \times [0,1] \to \mathbb{S}^n$ and regard it as a map $H\colon X \times \{0\} \cup A \times [0,1] \to \mathbb{S}^n$ whose restriction to $X \times \{0\}$ agrees with F. Regard H also as a map into $\mathbb{R}^{n+1}$. The Tietze–Urysohn extension theorem gives an extension $P\colon X \times [0,1] \to \mathbb{R}^{n+1}$ of H. Put $W = P^{-1}(\mathbb{R}^{n+1} \setminus \{0\})$ and define $Q\colon W \to \mathbb{S}^n$ by

$$Q(x,t) = \frac{P(x,t)}{\|P(x,t)\|}. \tag{2.52}$$

Then Q is continuous on W. Clearly, we observe that $X \times \{0\} \cup A \times [0,1] \subset W$ and $Q|_{X\times\{0\}\cup A\times[0,1]} = P$. This argument proves, without using the terminology, that $\mathbb{S}^n$ is an ANR for normal spaces. The space X will now be used to define an appropriate domain for this extension.

Let U be the set of all $p \in X$ for which there exists a neighborhood N of p such that $N \times [0,1] \subset W$. The tube lemma and $A \times [0,1] \subset W$ imply that $A \subset U$. By the normality of X, there exists a continuous map $u\colon X \to [0,1]$ such that $u(A) = 1$ and $u(X \setminus U) = \{0\}$. Define $L\colon X \times [0,1] \to \mathbb{S}^n$ by

$$L(x,t) = Q(x, t \cdot u(x)). \tag{2.53}$$

This map is continuous, the map $G(x) = L(x,1)$ extends g, and G is homotopic to F. □

Theorem 2.31. *Let X be a countably paracompact normal space. Then $\dim X \le n$ is equivalent to the following condition: for every closed subset A of X and every continuous map $f\colon A \to \mathbb{S}^n$, there exists a continuous map $F\colon X \to \mathbb{S}^n$ satisfying $F|_A = f$.*

Proof. Since they are homeomorphic, we identify $\mathbb{S}^n$ with $\mathrm{Bdry}\,\mathbb{I}^{n+1}$. First assume that $\dim X \le n$. Regard f as a map $f\colon A \to \mathrm{Bdry}\,\mathbb{I}^{n+1}$ and write $f = (f_1, \dots, f_{n+1})$. Define K_i and L_i by

$$K_i = \{\, x \in A \mid f_i(x) = -1 \,\}; \tag{2.54}$$

$$L_i = \{\, x \in A \mid f_i(x) = 1 \,\}. \tag{2.55}$$

Then K_i and L_i are closed subsets of X satisfying $K_i \cap L_i = \emptyset$ and

$$A = \bigcup_{i=1}^{n+1} (K_i \cup L_i). \tag{2.56}$$

Apply Proposition 2.25 to $\{K_i\}_{i=1}^{n+1}$ and $\{X \setminus L_i\}_{i=1}^{n+1}$. There exist open sets P_i and Q_i such that, for each $i \in \{1, \dots, n+1\}$, we have $K_i \subset P_i$, $\mathrm{CL}\,P_i \subset Q_i$, $\mathrm{CL}\,Q_i \subset X \setminus L_i$, and $\bigcap_{i=1}^{n+1} (\mathrm{CL}\,Q_i \setminus P_i) = \emptyset$. The Tietze–Urysohn extension theorem gives, for each i, a continuous map $u_i\colon X \to [-1,1]$ such that $u_i(\mathrm{CL}\,P_i) = -1$ and $u_i(X \setminus Q_i) = 1$. Since $u_i^{-1}(0) \subset Q_i \setminus \mathrm{CL}(P_i) \subset \mathrm{CL}(Q_i) \setminus P_i$, we have $\bigcap_{i=1}^{n+1} u_i^{-1}(\{0\}) = \emptyset$. Notice that $K_i \subset u_i^{-1}(\{-1\})$ and $L_i \subset u_i^{-1}(\{1\})$ for every $i \in \{1, \dots, n+1\}$.

Define $u\colon X \to [-1,1]^{n+1}$ by $u = (u_1, \dots, u_{n+1})$. By $A = \bigcup_{i=1}^{n+1} (K_i \cup L_i)$ and the properties of the u_i, if $x \in A$, then $u(x)$ belongs to $\mathrm{Bdry}\,\mathbb{I}^{n+1}$. Put $v = u|_A\colon A \to \mathrm{Bdry}\,\mathbb{I}^{n+1}$ and define $h\colon A \times [0,1] \to \mathbb{R}^{n+1}$ by

$$h(x,t) = (1-t)v(x) + tf(x). \tag{2.57}$$

It follows from $A = \bigcup_{i=1}^{n+1} (K_i \cup L_i)$ and the definitions of K_i, L_i, and u_i that $h(x,t)$ always has a coordinate whose absolute value is 1. For example, if $x \in K_i$, then $u_i(x) = f_i(x) = -1$, and hence the i-th coordinate of $h(x,t)$ is -1. Thus $\|h(x,t)\|_\infty \neq 0$, where $\|*\|_\infty$ denotes the ℓ^∞ -norm. We can therefore define a continuous map $H\colon A \times [0,1] \to \operatorname{Bdry} \mathbb{I}^{n+1}$ by

$$H(x,t) = \frac{h(x,t)}{\|h(x,t)\|_\infty}. \tag{2.58}$$

Notice that $H(x,0) = v(x)$ and $H(x,1) = f(x)$. Thus H is a homotopy from v to f. Since v has the continuous extension $u/\|u\|_\infty$ to all of X, Borsuk's homotopy extension theorem (Theorem 2.30) implies that f has an extension $F\colon X \to \operatorname{Bdry} \mathbb{I}^{n+1}$.

Conversely, assume that the extension problem always has a solution. Let $\{U_1, \ldots, U_{n+1}\}$ be an arbitrary family of $(n+1)$-many open subsets of X, and let $\{F_1, \ldots, F_{n+1}\}$ be a family of closed subsets satisfying $F_i \subset U_i$. Put $H_i = X \setminus U_i$ and $A = \bigcup_{i=1}^{n+1}(F_i \cup H_i)$. For each index, choose $u_i\colon A \to [-1,1]$ such that $u_i(F_i) = \{-1\}$ and $u_i(H_i) = \{1\}$. Then $u = (u_1, \ldots, u_{n+1})$ is a map into $\operatorname{Bdry} \mathbb{I}^{n+1}$. By the hypothesis, there exists a continuous map $g = (g_1, \ldots, g_{n+1})\colon X \to \operatorname{Bdry} \mathbb{I}^{n+1}$ such that $g|_A = u$. Put $V_i = g_i^{-1}([-1,0))$. Then $\operatorname{CL} V_i \subset g_i^{-1}([-1,0])$ and $F_i \subset V_i \subset \operatorname{CL} V_i \subset U_i$ for every $i \in \{1, \ldots, n+1\}$. Moreover, $\operatorname{Bdry} V_i \subset g_i^{-1}(\{0\})$. Since g takes values in $\operatorname{Bdry} \mathbb{I}^{n+1}$, it follows that $\bigcap_{i=1}^{n+1} \operatorname{Bdry} V_i = \emptyset$. Therefore, Proposition 2.25 gives $\dim X \leq n$. □

Remark 2.1. In fact, Theorem 2.30 remains valid without the assumption that X is countably paracompact ([10]). Consequently, Theorem 2.31 also holds without this assumption. In this manuscript, however, these results are used only when X is metrizable, so including countable paracompactness does not affect the development of the theory in this manuscript.

3. The fixed point theorem

In this section, we clarify the relationship between the dimension of Euclidean space and the fixed point theorem.

3.1. Statements equivalent to the fixed point theorem.

With the proof of Brouwer's fixed point theorem in mind, we first introduce several statements equivalent to it.

We begin with the following preliminary lemma.

Lemma 3.1. *Let A be a closed subset of $[-1,1]$ that separates $\{-1\}$ and $\{1\}$. Write $[-1,1] \setminus A = U \coprod V$, where $1 \in V$. Then, for every element x of V,*

$$|x - d(x,A)| \leq 1.$$

Proof. First assume that $0 \leq x - d(x,A)$. Since $0 \leq d(x,A)$, we obtain $|x - d(x,A)| \leq x - d(x,A) \leq x \leq 1$. Next assume that $0 \leq d(x,A) - x$. For $c = \min A$, the fact that A separates $\{-1\}$ and $\{1\}$ implies $-1 < c$. The definitions of U and V, together with $1 \in V$, also imply $c < x$.

Hence $d(x, A) \leq |x-c| = x-c$. Therefore, $|x-d(x, A)| = d(x, A)-x \leq x - c - x = -c < 1$. $\square$

The following theorem collects several well-known statements equivalent to Brouwer's fixed point theorem.

Theorem 3.2. *For $n \in \mathbb{Z}_{\geq 1}$, the following statements are equivalent.*

1. *(No-retraction theorem) There is no retraction $r \colon \mathbb{D}^n \to \mathbb{S}^{n-1}$.*
2. *(Brouwer's fixed point theorem) Every continuous map $f \colon \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.*
3. *If a family of closed sets $C_i \subset \mathbb{I}^n$, indexed by $i \in \{1, \dots, n\}$, is such that each member separates $\mathbb{I}^{n,i}_{+}$ and $\mathbb{I}^{n,i}_{-}$, then $\bigcap_{i=1}^{n} C_i \neq \emptyset$.*
4. *(The Poincaré–Miranda theorem) Assume that a family of continuous maps $f_i \colon \mathbb{I}^n \to \mathbb{R}$, indexed by $i \in \{1, \dots, n\}$, satisfies $f_i(\mathbb{I}^{n,i}_{+}) \subset [0, \infty)$ and $f_i(\mathbb{I}^{n,i}_{-}) \subset (-\infty, 0]$. Then there exists $p \in \mathbb{I}^n$ such that $f_i(p) = 0$. In other words, the maps have a common zero.*

Proof. Proof of $[(1) \Longrightarrow (2)]$: Suppose that there exists a continuous map $f \colon \mathbb{D}^n \to \mathbb{D}^n$ with no fixed point. Thus, $f(x) \neq x$ for every $x \in \mathbb{D}^n$. Define $e(x) = x - f(x)$. Then $e \colon \mathbb{D}^n \to \mathbb{R}^n$ is continuous. Define also $L \colon \mathbb{D}^n \times \mathbb{R} \to \mathbb{R}^n$ by $L(x, t) = f(x) + te(x)$. This map is continuous as well.

Consider the quadratic equation $\|L(x, t)\|^2 = 1$ in t, that is,

$$\|e(x)\|^2 t^2 + 2\langle f(x), e(x)\rangle t + \|f(x)\|^2 - 1. \tag{3.1}$$

Notice that its coefficients are functions of x. Here we regard it as an equation in t. The solutions of (3.1) depend on x. The two solutions are

$$\begin{aligned} &\frac{1}{\|e(x)\|^2}\Big(- \langle f(x), e(x)\rangle \\ &\qquad \pm \sqrt{\langle f(x), e(x)\rangle^2 - \|e(x)\|^2(\|f(x)\|^2 - 1)}\Big). \end{aligned} \tag{3.2}$$

Since $\|f(x)\| \leq 1$, we have $-\|e(x)\|^2(\|f(x)\|^2 - 1) \geq 0$. Consequently,

$$\sqrt{\langle f(x), e(x)\rangle^2 - \|e(x)\|^2(\|f(x)\|^2 - 1)} \geq |\langle f(x), e(x)\rangle|. \tag{3.3}$$

Thus (3.1) has real solutions, one nonnegative and one nonpositive. Denote the nonnegative solution by $T(x)$. The value $T(x)$ is given explicitly by

$$\begin{aligned} &T(x) = \\ &\frac{1}{\|e(x)\|^2}\Big(-\langle f(x), e(x)\rangle + \sqrt{\langle f(x), e(x)\rangle^2 - \|e(x)\|^2(\|f(x)\|^2 - 1)}\Big). \end{aligned} \tag{3.4}$$

This formula shows that $T \colon \mathbb{D}^n \to \mathbb{R}$ is continuous. If $x \in S^{n-1}$, then $L(x, 1) = x$, and hence $T(x) = 1$. Define $g \colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ by

$g(x) = L(x, T(x))$. This map is continuous, and when $x \in S^{n-1}$, we have $g(x) = L(x, T(x)) = L(x, 1) = x$. Thus g is a retraction, so $\mathbb{S}^{n-1}$ is a retract of $\mathbb{D}^n$. The existence of the continuous retraction $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ contradicts (1). Therefore (2) holds.

Proof of [(2) $\Longrightarrow$ (3)]: For every $i \in \{1, \ldots, n\}$, choose open subsets P_i and N_i of $\mathbb{I}^n$ such that $\mathbb{I}^n \setminus C_i = N_i \coprod P_i$, $\mathbb{I}^{n,i}_+ \subset P_i$, and $\mathbb{I}^{n,i}_- \subset N_i$. There may be more than one such pair, since the complement of C_i may have more than two connected components; choose any one of them. Define $m_i\colon \mathbb{I}^n \to \mathbb{R}$ by

$$m_i(x) = \begin{cases} d(x, C_i) & x \in C_i \cup P_i; \\ -d(x, C_i) & x \in C_i \cup N_i. \end{cases} \tag{3.5}$$

Both $C_i \cup P_i$ and $C_i \cup N_i$ are closed. On their intersection C_i, the two functions $d(x, C_i)$ and $-d(x, C_i)$ have the same value 0. They are continuous on $C_i \cup P_i$ and $C_i \cup N_i$, respectively. The pasting lemma therefore shows that m_i is well-defined and continuous.

Define $f_i\colon \mathbb{I}^n \to \mathbb{R}$ by

$$f_i(x) = x_i - m_i(x). \tag{3.6}$$

We show that $|f_i(x)| \leq 1$. We give the proof only for $x \in P_i$, as the other case is analogous. Here $f_i(x) = x_i - d(x, C_i)$. If $0 \leq x_i - d(x, C_i)$, then $0 \leq d(x, C_i)$, and hence

$$|f_i(x)| = x_i - d(x, C_i) \leq x_i \leq 1$$

as desired. Next consider the case $0 \leq d(x, C_i) - x_i$. Let

$$L = \{\, z \in \mathbb{I}^n \mid z_k = x_k, \text{ for all } k \text{ with } k \neq i \,\} \tag{3.7}$$

be the line through x perpendicular to $\mathbb{I}^{n,i}_+$ and $\mathbb{I}^{n,i}_-$. Let a be the unique point of $L \cap \mathbb{I}^{n,i}_+$, and let b be the unique point of $L \cap \mathbb{I}^{n,i}_-$. Thus the i-th coordinate of a is 1, and its other coordinates agree with those of x; the i-th coordinate of b is -1, and its other coordinates agree with those of x. The set $L \cap C_i$ separates a and b as a subset of L. Identifying L with $[-1, 1]$, we may apply Lemma 3.1 to obtain $|x_i - d(x, L \cap C_i)| \leq 1$. Now calculate $|f_i(x)|$:

$$\begin{aligned} |f_i(x)| &= d(x, C_i) - x_i \leq d(x, L \cap C_i) - x_i \\ &= |d(x, L \cap C_i) - x_i| \leq 1. \end{aligned} \tag{3.8}$$

This proves the desired inequality. Thus $|f_i(x)| \leq 1$ in all cases.

Define $f\colon \mathbb{I}^n \to \mathbb{I}^n$ by $f = (f_1, \ldots, f_n)$. The preceding inequality shows that the range of f is contained in $\mathbb{I}^n$. By (2), there exists a fixed point p. This point p satisfies, by the definition of f, we see that $d(p, C_i) = 0$. Thus $p \in \bigcap_{i=1}^n C_i$, so $\bigcap_{i=1}^n C_i \neq \emptyset$. Hence (3) follows.

Proof of [(3) $\Longrightarrow$ (4)]: For every $i \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}_{\geq 0}$, put

$$L_{i,k} = \{\, x \in \mathbb{I}^n \mid |x_i| \leq 1 - 2^{-k} \,\}. \tag{3.9}$$

Define $w_{i,k}\colon \mathbb{I}^n \to \mathbb{R}$ by

$$w_{i,k}(x) = x_i \cdot d(x, L_{i,k}). \tag{3.10}$$

Define $g_{i,k}\colon \mathbb{I}^n \to \mathbb{R}$ by $g_{i,k}(x) = f_i(x)+w_{i,k}(x)$. Then $g_{i,k}(\mathbb{I}^{n,i}_+) \subset (0,\infty)$ and $g_{i,k}(\mathbb{I}^{n,i}_-) \subset (-\infty, 0)$. Therefore, the set $C_{i,k} = g_{i,k}^{-1}(\{0\})$ is a closed set separating $\mathbb{I}^{n,i}_+$ and $\mathbb{I}^{n,i}_-$. By (3), we have $\bigcap_{i=1}^{n} C_{i,k} \neq \emptyset$. Choose p_k from this intersection. By compactness of $\mathbb{I}^n$, there exists a convergent subsequence $\{p_{\phi(k)}\}_{k\in\mathbb{Z}_{\geq 0}}$. Denote its limit by p. Since $g_{i,k}$ converges uniformly to f_i, we have $g_{i,\phi(k)}(p_{\phi(k)}) \to f_i(p)$ as $k \to \infty$. Consequently, $f_i(p) = 0$ for every $i \in \{1, \dots, n\}$. Thus (4) holds.

Proof of [(4) $\Longrightarrow$ (1)]: Identify $\mathbb{I}^n$ with $\mathbb{D}^n$, and identify $\mathbb{S}^{n-1}$ with $\operatorname{Bdry}\mathbb{I}^n$. Suppose, toward a contradiction, that there exists a retraction $r\colon \mathbb{I}^n \to \operatorname{Bdry}\mathbb{I}^n$. The map r has components r_i satisfies $r_i(\mathbb{I}^{n,i}_+) \subset [0,\infty)$ and $r_i(\mathbb{I}^{n,i}_-) \subset (-\infty, 0]$. By (4), there exists $p \in \mathbb{I}^n$ such that $r(p) = 0$. This contradicts the fact that the range of r is $\operatorname{Bdry}\mathbb{I}^n$. Therefore no such retraction exists, and (1) holds. □

3.2. **Proof of the fixed point theorem.** Theorem 3.2 establishes only the equivalence of several statements and does not by itself show that the fixed point theorem is true. The following known result shows that all the statements in Theorem 3.2 are indeed true.

Theorem 3.3. *For every $n \in \mathbb{Z}_{\geq 1}$, there is no retraction $r\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$.*

Proof. We first give a proof using homology. When $n = 1$, the nonexistence of a retraction follows from the intermediate value theorem. Henceforth assume that $1 < n$. Let $H_i(*)$ denote the homology functor. Suppose that a retraction as above exists. Let ι be the inclusion of $\mathbb{S}^{n-1}$ into $\mathbb{D}^n$. Since $r \circ \iota = 1_{\mathbb{S}^{n-1}}$, we obtain, in degree $n-1$,

$$H_{n-1}(r) \circ H_{n-1}(\iota) = H_{n-1}(1_{\mathbb{S}^{n-1}}) = 1_{H_{n-1}(\mathbb{S}^{n-1})}$$

Thus $H_{n-1}(\iota)\colon H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(\mathbb{D}^n)$ is injective. However, we have $H_{n-1}(\mathbb{D}^n) \cong \{0\}$ and $H_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$, so no such injective map $H_{n-1}(\iota)$ can exist. Therefore the retraction r does not exist. □

One can also prove analytically that $\mathbb{S}^{n-1}$ is not a retract of $\mathbb{D}^n$. We do so below using a statement about smooth retractions.

Proposition 3.4. *If there exists a retraction $f\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$, then there exists a smooth retraction $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$. Here, smoothness of g on $\mathbb{D}^n$ means that g is defined and smooth on an open set containing $\mathbb{D}^n$.*

Proof. Define $u(x) = f(x) - x$. Notice that $\|u(x)\| \leq 2$. Since $f|_{\mathbb{S}^{n-1}} = 1_{\mathbb{S}^{n-1}}$, we have $u|_{\mathbb{S}^{n-1}} = 0$. Hence there exists $\delta \in (3/4, 1)$ such that

$$\delta \leq \|x\| \leq 1 \Longrightarrow \|u(x)\| < 4^{-1} \tag{3.11}$$

By the Weierstrass approximation theorem, there also exists a map $P\colon \mathbb{R}^n \to \mathbb{R}^n$ whose components are polynomials and such that

$$\|x\| \le 1 \Longrightarrow \|P(x) - u(x)\| < 4^{-1}. \tag{3.12}$$

Choose a polynomial $Q\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$r \in [0,\delta] \Longrightarrow 3/4 \le Q(r^2) \le 1; \tag{3.13}$$

$$r \in [\delta,1] \Longrightarrow \|Q(r^2)\| \le 1; \tag{3.14}$$

$$Q(1) = 0. \tag{3.15}$$

Define $v(x) = x + Q(\|x\|^2)P(x)$. Then $v\colon \mathbb{D}^n \to \mathbb{R}^n$ is continuous and smooth on $\mathbb{D}^n$. If $\|x\| \in [0,\delta]$, we have

$$\begin{aligned}
\|v(x)\| &= \|x + Q(\|x\|^2)P(x)\| \\
&= \|x + f(x) - f(x) + Q(\|x\|^2)P(x)\| \\
&= \left\| \begin{array}{l} f(x) - (f(x) - x) + Q(\|x\|^2)(f(x) - x) \\ \quad - Q(\|x\|^2)(f(x) - x) + Q(\|x\|^2)P(x) \end{array} \right\| \\
&= \left\| \begin{array}{l} f(x) + Q(\|x\|^2)(P(x) - f(x) + x) \\ \quad + (Q(\|x\|^2) - 1)(f(x) - x) \end{array} \right\| \\
&\ge \|f(x)\| - |Q(\|x\|^2)|\|P(x) - (f(x) - x)\| \\
&\qquad - |1 - Q(\|x\|^2)|\|f(x) - x\|.
\end{aligned}$$

Now $\|P(x) - (f(x) - x)\| < 1/4$, $\|f(x)\| = 1$, and $|Q(\|x\|^2)| \le 1$. Moreover, if $\|x\| \in [0,\delta]$, then $3/4 \le Q(\|x\|^2) \le 1$, so $0 \le 1 - Q(\|x\|^2) \le 1/4$. Combining these observations with the preceding calculation gives

$$\begin{aligned}
&\|v(x)\| \ge \\
&\|f(x)\| - |Q(\|x\|^2)|\|P(x) - (f(x) - x)\| - |1 - Q(\|x\|^2)|\|f(x) - x\| \\
&\ge 1 - \frac{1}{4} - \frac{2}{4} = \frac{1}{4}.
\end{aligned}$$

Thus $\|v(x)\| \ge 1/4$ if $\|x\| \in [0,\delta]$.

On the other hand, if $\|x\| \in [\delta,1]$, then

$$\begin{aligned}
&\|v(x)\| = \|x + Q(\|x\|^2)P(x)\| \\
&= \|x + Q(\|x\|^2)P(x) + Q(\|x\|^2)(f(x) - x) - Q(\|x\|^2)(f(x) - x)\| \\
&\ge \|x + Q(\|x\|^2)(P(x) - (f(x) - x)) + Q(\|x\|^2)(f(x) - x)\| \\
&\ge \|x\| - |Q(\|x\|^2)|(\|P(x) - (f(x) - x)\| + \|f(x) - x\|).
\end{aligned}$$

Here $|Q(\|x\|^2)| \le 1$ and $\|P(x) - (f(x) - x)\| \le 1/4$. Since $\|x\| \in [\delta,1]$, we also have $\|f(x) - x\| \le 1/4$. Therefore,

$$\begin{aligned}
\|v(x)\| &\ge \|x\| - |Q(\|x\|^2)|(\|P(x) - (f(x) - x)\| + \|f(x) - x\|) \\
&\ge \delta - \left(\frac{1}{4} + \frac{1}{4}\right) \ge \frac{3}{4} - \frac{2}{4} = \frac{1}{4}.
\end{aligned}$$

Thus $\|v(x)\| \geq 1/4$ if $\|x\| \in [\delta, 1]$.

It follows that $\|v(x)\| \neq 0$ for every $x \in \mathbb{D}^n$. We may therefore define $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ by

$$g(x) = \frac{v(x)}{\|v(x)\|}. \tag{3.16}$$

This map is smooth on $\mathbb{D}^n$. If $x \in \mathbb{S}^{n-1}$, then $Q(\|x\|^2) = Q(1) = 0$, so $v(x) = x$ and hence $g(x) = x$. Thus $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ is the desired retraction. □

Theorem 3.5 ([9])**.** *There is no smooth map $f\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ such that $f|_{\mathbb{S}^{n-1}} = 1_{\mathbb{S}^{n-1}}$.*

Proof. Suppose that such a map exists, and derive a contradiction. Put $g(x) = f(x) - x$. Define $F\colon \mathbb{D}^n \times [0,1] \to \mathbb{D}^n$ by

$$F(x,t) = x + tg(x) = x + t(f(x) - x) = (1-t)x + tf(x). \tag{3.17}$$

This is a homotopy from $1_{\mathbb{D}^n}$ to f that fixes $\mathbb{S}^{n-1}$. To emphasize the homotopy parameter, we also write $F_t(x)$ for $F(x,t)$. The point $F(x,t)$ divides the line segment joining x and $f(x)$.

Since $\mathbb{D}^n$ is compact and g is smooth, $g(x) = f(x) - x$ is Lipschitz. Let $L > 0$ be a Lipschitz constant for g. If $F(x,t) = F(y,t)$, then $x - y = t(g(y) - g(x))$, and hence

$$\|x - y\| \leq t\|g(y) - g(x)\| \leq tL\|x - y\|. \tag{3.18}$$

Thus, when $0 \leq t < L^{-1}$, we see that $F_t(x)$ is injective.

The Jacobian matrix of $F(x,t)$ with respect to x satisfies

$$J_x(F_t(x)) = E_n + tJ_x(g(x)). \tag{3.19}$$

Since "det" is continuous, if t_0 is sufficiently small, then we see that $\det J_x(F(x,t)) > 0$ for $t \in [0, t_0]$. Choose it still smaller so that $t_0 < L^{-1}$. We show that, for $t \in [0, t_0]$, the map $F(x,t)$ is bijective as a function of x.

Let U be the interior of $\mathbb{D}^n$, that is, $U = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. By the inverse function theorem, $F_t(U)$ is open in $\mathbb{R}^n$. Choose any $a \notin F_t(U)$ in $\mathbb{D}^n$. It suffices to show that a has a preimage under F_t. Choose $b \in F_t(U)$ in $\mathbb{D}^n$. The point b will be used below. Since $F_t(U) \subset \mathbb{D}^n$, the segment $[a,b]$ is contained in $\mathbb{D}^n$. As $[a,b]$ is connected, $[a,b] \cap \mathrm{Bdry}(F_t(U)) \neq \emptyset$. Choose c from this intersection. Since $F_t(\mathbb{D}^n)$ is compact and hence closed,

$$\mathrm{Bdry}(F_t(U)) \subset \mathrm{CL}(F_t(U)) \subset F_t(\mathbb{D}^n). \tag{3.20}$$

Thus there exists $x \in \mathbb{D}^n$ such that $c = F_t(x)$. Since $c \notin F_t(U)$, we have $x \in \mathbb{S}^{n-1}$. Because F_t fixes $\mathbb{S}^{n-1}$, it follows that $c = F_t(x) = x$. Thus $c \in \mathbb{S}^{n-1}$. Since $F_t(U)$ is open in $\mathbb{R}^n$, we have $\mathrm{Bdry}(F_t(U)) \cap F_t(U) = \emptyset$. In particular, $c \neq b$. Since $c \in [a,b]$ and $c \in \mathbb{S}^{n-1}$, we obtain $a = c$, and hence $a = F_t(x)$. Consequently, the map F_t is surjective.

We next define $I(t)$ by

$$I(t) = \int_{\mathbb{D}^n} \det(J_x(F(x,t)))\, dx_1 dx_2 \cdots dx_n. \tag{3.21}$$

The quantity $I(t)$ will be used below. Since $F_t(\mathbb{D}^n) = \mathbb{D}^n$, the change-of-variables formula gives $I(t) = \mathcal{L}^n(\mathbb{D}^n) > 0$ for $t \in [0, t_0]$. Here $\mathcal{L}^n(\mathbb{D}^n)$ denotes the n-dimensional Lebesgue measure of $\mathbb{D}^n$. The function $I(t)$ is a polynomial in t. Since it is constant on $[0, t_0]$, it must also be constant on $[0, 1]$. Thus, for every $t \in [0, 1]$,

$$I(t) = \mathcal{L}^n(\mathbb{D}^n) > 0. \tag{3.22}$$

On the other hand, $F_1(x) = f(x) \in \mathbb{S}^{n-1}$, so $\langle f, f \rangle = 1$. Differentiating with respect to x_i gives

$$\left\langle \frac{\partial f}{\partial x_i}, f \right\rangle = 0. \tag{3.23}$$

Thus $f(x)$ is an eigenvector of the transpose of $J_x(f)(x)$ with eigenvalue 0. A matrix with eigenvalue 0 has determinant 0, so $\det(J_x(f)(x)) = 0$. This implies $I(1) = 0$, contradicting (3.22). □

3.3. **Determination of the dimension of Euclidean spaces.** We now determine the dimension of Euclidean spaces.

Theorem 3.6. *Let $n \in \mathbb{Z}_{\geq 1}$ and $k \in \{0, \ldots, n\}$. Then we have* $\dim\mathbb{D}^n = \dim\mathbb{R}^n = n$. *Moreover, we also have* $\dim\mathcal{M}(n,k) = k$ *and* $\dim\mathcal{L}(n,k) = n - k$.

Proof. We have already established the fixed point theorem. Theorem 3.2 and Proposition 2.26 therefore imply $n - 1 < \dim\mathbb{I}^n$. Together with 2.13, this gives $\dim\mathbb{R}^n = \dim\mathbb{D}^n = n$. It also follows that $\dim\mathcal{M}(n,k) = k$ and $\dim\mathcal{L}(n,k) = n - k$. □

Corollary 3.7. *Let $n, m \in \mathbb{Z}_{\geq 1}$. If $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.*

Corollary 3.8. *Let $n, k \in \mathbb{Z}_{\geq 0}$. Assume that subsets $A_1, \ldots, A_k$ of $\mathbb{R}^n$ satisfy $\mathbb{D}\mathrm{im} A_i \leq 0$ and $\mathbb{R}^n = \bigcup_{i=1}^k A_i$. Then $n + 1 \leq k$.*

In view of their historical development and the methods used in their proofs, Theorem 3.2, Theorem 4.2, and Corollary 3.7 may be regarded as equivalent in a meaningful sense. Of course, any two true statements are logically equivalent, but these results seem to be equivalent in a deeper way. Inspired by Terence Tao's proof of the invariance of domain theorem [11], the paper [6] appears to examine this sense of equivalence from the viewpoint of reverse mathematics. The author is not sufficiently familiar with reverse mathematics to discuss the matter further.

4. The invariance of domain theorem

In this section, we prove Brouwer's invariance of domain theorem as an application of dimension theory. It states that the image of an open subset U of an n-dimensional space under a continuous injective map into n-dimensional Euclidean space is open.

4.1. **Proof of the invariance of domain theorem.**

The following theorem shows that whether a point of a compact subset K of Euclidean space is a boundary point can be characterized solely in terms of the intrinsic topology of K. This is the key to the invariance of domain theorem.

Theorem 4.1. *Let $n \in \mathbb{Z}_{\geq 1}$, and K be a compact subset of $\mathbb{R}^n$. Then $p \in \operatorname{Bdry}_{\mathbb{R}^n} K$ if and only if, for every neighborhood N of p in K, there exists an open neighborhood U of p in K such that $U \subset N$ and every continuous map $f\colon K \setminus U \to \mathbb{S}^{n-1}$ has a continuous extension $g\colon K \to \mathbb{S}^{n-1}$.*

Proof. First assume that $p \in \operatorname{Bdry} K$ and derive the stated condition. Let N be an arbitrary neighborhood of p. Choose an open ball B centered at p with sufficiently small radius that $K \cap B \subset N$. We show that $U = K \cap B$ has the required property. Put $L = \operatorname{Bdry} B$. Then L is homeomorphic to $\mathbb{S}^{n-1}$. Consider an arbitrary continuous map $f\colon K \setminus U \to \mathbb{S}^{n-1}$. The set $K \setminus U$ is closed in K and hence is also closed in $\mathbb{R}^n$. Since $\dim L \leq n-1$, we have $\dim(K \cap L) \leq n-1$. Notice that $L \cap K \subset K \setminus U$. By Theorem 2.31, the continuous map $f|_{L\cap K}\colon L \cap K \to \mathbb{S}^{n-1}$ has a continuous extension $u\colon L \to \mathbb{S}^{n-1}$. The intersection of L and $K \setminus U$ is $L \cap K$, and u and f agree there. The pasting lemma therefore gives a continuous map $h\colon (K\setminus U)\cup L \to \mathbb{S}^{n-1}$.

Since p is a boundary point of K, we can choose a point q of B that does not belong to K. For every $x \in K$, let $b(x)$ be the intersection of L with the line extended from q through $x \in K$. Then $b\colon K \to L$ is continuous. Define $g\colon K \to \mathbb{S}^{n-1}$ by

$$g(x) = \begin{cases} h(b(x)) & x \in U \cup (L \cap K); \\ f(x) & x \in K \setminus U. \end{cases}$$

The two functions $h(b(x))$ and $f(x)$ appearing in this definition agree on $L \cap K$. They therefore paste together to give a continuous map g, which is clearly an extension of f.

Conversely, assume that $p \in K$ satisfies the condition in the theorem. Suppose, toward a contradiction, that $p \in K$ is an interior point of K. Choose an open ball B centered at p with sufficiently small radius that $\operatorname{CL} B \subset K$. By the hypothesis, there exists an open neighborhood U of p such that $U \subset B$ and every continuous map $f\colon K \setminus U \to \mathbb{S}^{n-1}$ extends over K. Define a map $f\colon K \setminus U \to \mathbb{S}^{n-1}$ as follows. Extend the line from p through $x \in K \setminus U$, and let $f(x)$ be its intersection with $\operatorname{Bdry} B$. This is continuous, and since $\operatorname{Bdry} B$ is homeomorphic to $\mathbb{S}^{n-1}$, we obtain a continuous map $f\colon K \setminus U \to \mathbb{S}^{n-1}$. The map f is the radial map just described. Notice that f is the identity on $\operatorname{Bdry} B$. The condition in the theorem gives a continuous map $g\colon K \to \mathbb{S}^{n-1}$. We now restrict g to the closed ball. Recall that $\operatorname{CL} B \subset K$. Consider $g|_{\operatorname{CL} B}$. The restriction $g|_{\operatorname{CL} B}\colon \operatorname{CL} B \to \operatorname{Bdry} B$ is then a retraction from $\operatorname{CL} B$ onto $\operatorname{Bdry} B$. This contradicts the no-retraction theorem (Theorem 3.2). Therefore $p \in K$ must be a boundary point. □

We now prove the invariance of domain theorem.

Theorem 4.2 (Invariance of domain)**.** *Let $n \in \mathbb{Z}_{\geq 1}$, let U be an open subset of $\mathbb{R}^n$, and let $f\colon U \to \mathbb{R}^n$ be continuous and injective. Then $f(U)$ is open in $\mathbb{R}^n$.*

Proof. It suffices to prove that $f(p)$ is an interior point of $f(U)$ for every $p \in U$. Choose a sufficiently small neighborhood N of p such that $\operatorname{CL} N$ is compact and $\operatorname{CL} N \subset U$. The restriction $f|_{\operatorname{CL} N}$ is a continuous injection from a compact space into a Hausdorff space, and hence is a topological embedding. By Theorem 4.1, whether p is a boundary point is characterized by an intrinsic topological condition on $\operatorname{CL} N$. Since p is an interior point of $\operatorname{CL} N$, the point $f(p)$ is an interior point of $f(\operatorname{CL} N)$. In particular, $f(p)$ is an interior point of $f(U)$. □

Corollary 4.3. *Let M and N be topological manifolds with boundary of the same dimension, and let $f\colon M \to N$ be a homeomorphism. Then f maps ∂M onto ∂N.*

Proof. If f maps some $p \in \partial M$ into $N \setminus \partial N$, then, by considering f^{-1}, we obtain locally a map carrying an interior point of Euclidean space to p. But p is a boundary point of a half-space, contradicting the invariance of domain theorem. □

5. n-dimensional subsets of n-dimensional Euclidean space

As an application of dimension theory, we prove in this section that an n-dimensional subspace of n-dimensional Euclidean space has an interior point.

We first prove that Euclidean spaces have the property known as countable dense homogeneity. For the one-dimensional case, we need to study the order structure of the rational numbers.

5.1. **The order-theoretic characterization of the rationals and countable dense homogeneity of the line.** We begin with the definition of a linear order.

Definition 5.1 (Linear order)**.** A binary relation $\leq$ on a set X is called a linear order if it satisfies the following conditions.

(1) For every $a \in X$, we have $a \leq a$.
(2) For every $a, b \in X$, if $a \leq b$ and $b \leq a$, then $a = b$.
(3) For every $a, b, c \in X$, if $a \leq b$ and $b \leq c$, then $a \leq c$.
(4) For every $a, b \in X$, either $a \leq b$ or $b \leq a$.

We also define $a < b$ to mean that $a \leq b$ and $a \neq b$.

Definition 5.2 (Dense order)**.** A linearly ordered set $(X, \leq)$ is said to be **densely ordered** if, whenever $x < y$ for $x, y \in X$, there exists $s \in X$ such that $x < s < y$.

We sometimes write $\leq_X$ for the order on X, but omit the subscript and simply write $\leq$ when the context is clear.

Lemma 5.1. *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be countable densely ordered sets having neither a greatest nor a least element. Let $S \subset X$ and $T \subset Y$ be finite subsets, and assume that an order isomorphism $h\colon S \to T$ identifies S with T. Then, for every $x \in X \setminus S$, there exists $y \in Y \setminus T$ such that h extends to an order isomorphism $H\colon S \cup \{x\} \to T \cup \{y\}$.*

Proof. Without loss of generality, enumerate $S = \{s_i\}_{i=1}^n$ and $T = \{t_j\}_{j=1}^n$ so that

$$s_1 < s_2 < \cdots < s_n; \tag{5.1}$$

$$h(s_i) = t_i. \tag{5.2}$$

There are three cases.

Case 1. [If $s_i < x < s_{i+1}$ for some i]: Since Y is densely ordered, choose y such that $t_i < y < t_{i+1}$. Defining $H(x) = y$ and $H(s_i) = t_i$ gives the required map H.

Case 2. [If $s_n < x$]: Since Y is densely ordered and has no greatest element, choose $y \in Y$ such that $t_n < y$. Again define $H(x) = y$ and $H(s_i) = t_i$.

Case 3. [If $x < s_1$]: This is analogous to the preceding case, using the absence of a least element. □

For a map $f\colon A \to B$, define $\mathrm{dom} f = A$ and $\mathrm{cod} f = f(A)$.

Theorem 5.2. *Any two countable densely ordered sets having neither a greatest nor a least element are order-isomorphic. In particular, each is order-isomorphic to the rational numbers.*

Proof. We use the standard back-and-forth argument.

Let $(A, \leq_A)$ and $(B, \leq_B)$ be countable densely ordered sets with no greatest or least element. Write $A = \{a_i\}_{i \in \mathbb{Z}_{\geq 0}}, B = \{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$. Using these enumerations, inductively construct a sequence of extensions $\{f_n\}_{n \in \mathbb{Z}_{\geq 0}}$ satisfying the following conditions.

(1) f_n is an order-preserving bijection.
(2) $\mathrm{dom} f_n$ is finite and contains $\{a_0, \dots, a_n\}$.
(3) $\mathrm{cod} f_n$ is finite and contains $\{b_0, \dots, b_n\}$.

Step 0 (definition of f_0): Define f_0 to be the map $f_0 \colon \{a_0\} \to \{b_0\}$ given by $f_0(a_0) = b_0$.

Step 1 (construction of f_{n+1} from f_n): First define an auxiliary order isomorphism g_n. If $a_{n+1} \in \mathrm{dom}(f_n)$, put $g_n = f_n$. If $a_{n+1} \notin \mathrm{dom}(f_n)$, apply Lemma 5.1 to f_n, a_{n+1}, $\mathrm{cod}(f_n), \mathrm{dom}(f_n)$ and let g_n be an order-isomorphic extension of f_n to $\mathrm{dom}(f_n) \cup \{a_{n+1}\}$.

Next extend g_n^{-1} to define another auxiliary order isomorphism h_n. If $b_{n+1} \in \mathrm{dom}(g_n^{-1}) = \mathrm{cod}(g_n)$, put $h_n = g_n^{-1}$. If $b_{n+1} \notin \mathrm{dom}(g_n^{-1}) = \mathrm{cod}(g_n)$, apply Lemma 5.1 to g_n^{-1} and b_{n+1}, and let h_n be an extension of g_n^{-1} to $\mathrm{dom}(g_n^{-1}) \cup \{b_{n+1}\}$. Finally, define $f_{n+1} = h_n^{-1}$. All inductive conditions are then satisfied.

Step 3: Using the sequence of order isomorphisms $\{f_n\}_{n \in \mathbb{Z}_{\geq 0}}$, define $F \colon A \to B$ by $F(a_n) = f_n(a_n)$. Since $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of extensions, this definition is well-defined.

Step 4: We show that F is an order isomorphism. First let $x, y \in A$ and assume that $x < y$. We show that $F(x) < F(y)$, which gives injectivity and preservation of order. For sufficiently large n, we have $x, y \in \mathrm{dom}(f_n)$. By the definition of F and the fact that $\{f_n\}_{n \in \mathbb{Z}_{\geq 0}}$ is a sequence of extensions, $F(x) = f_n(x)$ and $F(y) = f_n(y)$. Since f_n is an order isomorphism, we immediately obtain $F(x) < F(y)$.

It remains to prove surjectivity. For every $y \in B$, we have $y \in \mathrm{cod}(f_n)$ for all sufficiently large n. Since f_n is an order isomorphism, there exists $x \in \mathrm{dom}(f_n)$ such that $f_n(x) = y$. Thus $F(x) = y$, so F is surjective. □

Remark 5.1. The construction alternates between A and B when defining g_n and h_n. This is the reason for the name “back-and-forth argument.”

Example 5.1. The subset $(0, 1) \cap \mathbb{Q}$ of $\mathbb{R}$ is order-isomorphic to $\mathbb{Q}$.

Example 5.2. The set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is order-isomorphic to $\mathbb{Q}$.

Example 5.3. The set of numbers having a terminating binary expansion, namely $\left\{\frac{m}{2^n} \;\middle|\; n, m \in \mathbb{Z}\right\}$, is order-isomorphic to $\mathbb{Q}$.

Lemma 5.3. *Let A be a countable dense subset of $\mathbb{R}$, and let $S \subset A$ have no greatest element. Assume that whenever $u, v \in A$, $v \in S$, and $u < v$, we have $u \in S$. Then there exists $y \in \mathbb{R}$ such that $S = A \cap (-\infty, y)$. Moreover, $A \setminus S = A \cap [y, \infty)$.*

Proof. Take $y = \sup S$. □

Theorem 5.4. *Let A and B be countable dense subsets of $\mathbb{R}$. Then an order isomorphism $f \colon A \to B$ extends to a homeomorphism $f \colon \mathbb{R} \to \mathbb{R}$ satisfying $f(A) = B$.*

Proof. Let A and B be countable dense subsets of $\mathbb{R}$. They satisfy the hypotheses of Theorem 5.2; hence they are order-isomorphic. Let

$f\colon A \to B$ be such an order isomorphism. Enumerate A and B so that the map $f\colon A \to B$ defined by $f(a_i) = b_i$ is an order isomorphism.

We extend this map to all of $\mathbb{R}$. For $x \in \mathbb{R}$, define $G_x = (-\infty, x)$ and $H_x = [x, \infty)$, and put $S_x = f(A \cap G_x)$, and $T_x = f(A \cap H_x)$. Since f preserves order, S_x has the following downward-closed property: if u, v satisfy $u < v$ and $v \in S_x$, then $u \in S_x$. Moreover, the set S_x has no greatest element. Lemma 5.3 therefore gives $y \in \mathbb{R}$ such that $S_x = B \cap (-\infty, y)$. Similarly, $T_x = B \cap [y, \infty)$. Define $f(x) = y$. Notice that if $x = a_i$, then $y = b_i$.

We prove that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. First we show that f preserves order. Assume $x < y$. By density of A, there exists $a_n \in A$ such that $x < a_n < y$. Then $b_n \notin S_{f(x)}$ and $b_n \in S_{f(y)}$. In particular, $b_n \in T_{f(x)}$, so $f(x) \leq b_n < f(y)$. Thus f preserves order and is, in particular, injective.

To prove continuity, fix $\epsilon > 0$ and $x \in \mathbb{R}$. By density of B, there exist b_N and b_M such that

$$f(x) - \epsilon < b_N < f(x) < b_M < f(x) + \epsilon. \tag{5.3}$$

Since f preserves order, $a_N < x < a_M$. Thus (a_N, a_M) is an open neighborhood of x. For every $z \in (a_N, a_M)$, order preservation and (5.3) give $|f(z) - f(x)| < \epsilon$. Therefore f is continuous.

Extend $f^{-1}\colon B \to A$ in the same manner to obtain a continuous map $g\colon \mathbb{R} \to \mathbb{R}$. For every $n \in \mathbb{Z}_{\geq 0}$, we have $g(f(a_n)) = a_n$ and $f(g(b_n)) = b_n$. By continuity of f and g, it follows that, for every $x \in \mathbb{R}$, $f(g(x)) = x$ and $g(f(x)) = x$. Thus f and g are inverses. Hence f is a homeomorphism. By definition, $f(A) = B$, which proves the theorem. $\square$

5.2. **Countable dense homogeneity of Euclidean spaces: the general case.** We now establish countable dense homogeneity for Euclidean spaces. Namely, given countable dense subsets A and B of $\mathbb{R}^n$, we prove that there exists a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(A) = B$. We begin with an identity theorem for polynomials.

Theorem 5.5 (Identity theorem for polynomials)**.** *Let p be a real polynomial in n variables. If its zero set Z has an interior point, then p is identically 0. Equivalently, if p is not the constant polynomial 0, then the set of all $x \in \mathbb{R}^n$ such that $p(x) \neq 0$ is dense in $\mathbb{R}^n$.*

Proof. For $n = 1$, the result follows because a polynomial of degree k has at most k zeros. Now consider general n. Let a be an interior point of Z. For sufficiently small $r \in \mathbb{R}$, the open ball $B = \{\, x \in \mathbb{R}^n \mid \|x - a\| < r \,\}$ is contained in Z. Fix $u \in B$ and define $q\colon \mathbb{R} \to \mathbb{R}$ by $q(t) = p((1-t)a + tu)$. Then q is a polynomial in t. Since p vanishes on B, we have $q(t) = 0$ for $t \in (-1, 1)$. The one-variable case implies that q vanishes identically. Since $u \in B$ was arbitrary, p vanishes identically on the whole space. $\square$

Lemma 5.6. *Let* $n \in \mathbb{Z}_{\geq 1}$. *For* $v = (v_1, \ldots, v_n) \in \mathbb{R}^{n \times n}$, $i \in \{1, \ldots, n\}$, *and* $a \in \mathbb{R}^n$, *let* $\mathrm{Sp}(v, i, a)$ *be the linear subspace of* $\mathbb{R}^n$ *generated by* $\{v_k\}_{k \neq i}$ *and* a. *Then*

$$T_{a,i} = \{ v = (v_1, \ldots, v_n) \in \mathbb{R}^{n \times n} \mid \operatorname{rank} \mathrm{Sp}(v, i, a) = n \}$$

is a dense open subset of $\mathbb{R}^{n \times n}$.

Proof. For $v = (v_1, \ldots, v_n) \in \mathbb{R}^{n \times n}$, let $f(v)$ be the determinant of the matrix obtained from $(v_1, \ldots, v_n)$ by replacing its i-th column by a. Then f is a polynomial on $\mathbb{R}^{n \times n}$ and is not identically 0. Thus the set of v for which $f(v) \neq 0$ is dense, by Theorem 5.5, and is also open. Since $\operatorname{rank} \mathrm{Sp}(v, i, a) = n$ is equivalent to $f(v) \neq 0$, the proof is complete. $\square$

Let $\mathrm{GL}(n)$ denote the set of all invertible $n \times n$ matrices. Notice that a tuple $v = (v_1, \ldots, v_n) \in \mathrm{GL}(n)$ is a basis of $\mathbb{R}^n$. For such a tuple and $a \in \mathbb{R}^n$, let $\mathrm{pr}_{v,i}(a)$ denote the i-th coordinate of a with respect to the basis $v_1, \ldots, v_n$. Thus $a = \sum_{i=1}^{n} \mathrm{pr}_{v,i}(a) v_i$.

Theorem 5.7. *Let* $n \in \mathbb{Z}_{>0}$, *and let* S *be a countable subset of* $\mathbb{R}^n$ *such that* $0 \notin S$. *Then there exists a basis* $v = (v_1, \ldots, v_n) \in \mathbb{R}^{n \times n}$ *such that* $\mathrm{pr}_i(x) \neq 0$ *for every* $x \in S$.

Proof. Applying Theorem 5.5 to $\mathbb{R}^{n \times n}$ and det shows that $\mathrm{GL}(n)$ is a dense open subset of $\mathbb{R}^{n \times n}$. Define T by

$$T = \mathrm{GL}(n) \cap \bigcap_{a \in S} \bigcap_{i=1}^{n} T_{a,i} \tag{5.4}$$

Each $T_{a,i}$ is dense and open, so the Baire category theorem shows that T is dense in $\mathbb{R}^{n \times n}$. Choose $v = (v_1, \ldots, v_n) \in T$. This is the required basis. Indeed, otherwise there would exist i and $a \in S$ such that $\mathrm{pr}_{v,i}(a) = 0$. This means that a belongs to the linear span of $\{v_k\}_{k \neq i}$, which implies $\operatorname{rank} \mathrm{Sp}(v, i, a) < n$. This contradicts $v \in T_{a,i}$. $\square$

Below we abbreviate $\mathrm{pr}_{v,i}(x)$ as x_i or $(x)_i$.

Definition 5.3. A pair $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$ is said to be **in coordinatewise general position** if, for every $i \in \{1, \ldots, n\}$,

$$(x_1)_i - (x_2)_i \neq 0$$

A set $A \subset \mathbb{R}^n$ is said to be in coordinatewise general position if every distinct pair $(x, y) \in A \times A$ is in coordinatewise general position. Following [5], this condition is called general position; we add "coordinatewise" to distinguish it from the terminology used later.

Lemma 5.8. *Let* A *be a countable subset of* $\mathbb{R}^n$. *Then there exists a basis* $(v_1, v_2, \ldots, v_n)$ *of* $\mathbb{R}^n$ *such that, in the associated coordinate system,* A *is in coordinatewise general position.*

Proof. Apply Theorem 5.7 to $S = \{ x - y \mid x, y \in A \} \setminus \{0\}$. $\square$

Definition 5.4. Two pairs $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n$, in coordinatewise general position, are said to be **similarly placed** if, for every $i \in \{1, \ldots, n\}$, the numbers $(x_1)_i - (x_2)_i$ and $(y_1)_i - (y_2)_i$ have the same sign.

Two indexed at most countable sets, of the same cardinality when finite, $A = \{a_i\}_{i \in \mathbb{Z}_{\geq 0}}, B = \{b_i\}_{i \in \mathbb{Z}_{\geq 0}} \subset \mathbb{R}^n$, are said to be **similarly placed** if, for every $(i, j) \in \mathbb{Z}^2_{\geq 0}$, the pairs (a_i, a_j) and (b_i, b_j) are similarly placed.

Proposition 5.9. *Let $A = \{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$ and $B = \{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be countable dense subsets of $\mathbb{R}^n$, each in coordinatewise general position. They can be enumerated so that A and B are similarly placed.*

Proof. Inductively we construct new enumerations $\{c_i\}_{i \in \mathbb{Z}_{\geq 0}} = A$ and $\{d_i\}_{i \in \mathbb{Z}_{\geq 0}} = B$ as follows. First set $c_0 = a_0$, $d_0 = b_0$, and $d_1 = b_1$. Let i be the least natural number for which (c_0, a_i) and (d_0, d_1) are similarly placed, and put $c_1 = a_i$. Such an i exists because A is dense.

Assume generally that the terms through odd index $2j+1$ have been constructed, namely $c_0, c_1, \ldots, c_{2j+1}$ and $d_0, d_1, \ldots, d_{2j+1}$. Construct c_{2j+2} and d_{2j+2} as follows. Let $s \in \mathbb{Z}_{\geq 0}$ be the least index i such that a_i does not belong to $c_0, c_1, \ldots, c_{2j+1}$, and define $c_{2j+2} = a_s$. Next, let t be the least index such that $c_0, c_1, \ldots, c_{2j}, c_{2j+1}$ and $d_0, d_1, \ldots, d_{2j}, b_t$ are similarly placed, and define $d_{2j+2} = b_t$.

Now construct c_{2j+3} and d_{2j+3}. Let p be the least index such that b_p does not belong to $d_0, d_1, \ldots, d_{2j+2}$, and define $d_{2j+3} = b_p$. Next, let q be the least index for which $c_0, c_1, \ldots, c_{2j+2}, a_q$ and $d_0, d_1, \ldots, d_{2j+2}, d_{2j+3}$ are similarly placed, and define $c_{2j+3} = a_q$. This inductive construction is possible because both A and B are dense. $\square$

Theorem 5.10. *Let A and B be countable dense subsets of $\mathbb{R}^n$. Then there exists a homeomorphism $f \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(A) = B$.*

Proof. After changing bases and enumerations, we may assume that $A = \{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$ and $B = \{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ are similarly placed and both are in coordinatewise general position. Write $a_i = (a_{i,1}, \ldots, a_{i,n})$ and $b_i = (b_{i,1}, \ldots, b_{i,n})$. Because the coordinates were chosen so that A and B are in coordinatewise general position, if $i \neq j$, then, for every $k \in \{1, \ldots, n\}$, we have $a_{i,k} \neq a_{j,k}$ and $b_{i,k} \neq b_{j,k}$.

For $k \in \{1, \ldots, n\}$, put $A_k = \pi_k(A)$ and $B_k = \pi_k(B)$, where π_k is projection onto the k-th coordinate. The sets A_k and B_k are countable dense subsets of $\mathbb{R}$. Since their enumerations are similarly placed, the map defined by $f_k(a_{i,k}) = b_{i,k}$ is an order isomorphism. Without coordinatewise general position, indices could collapse, so this formula might fail to define a map, let alone an order isomorphism.

By Theorem 5.4, f_k extends to a homeomorphism $f_k \colon \mathbb{R} \to \mathbb{R}$ such that $f_k(A_k) = B_k$. Define $f \colon \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = (f_1(x_1), \ldots, f_n(x_n))$. For $a_i = (a_{i,1}, \ldots, a_{i,n})$, we have $f(a_i) = (f_1(a_{i,1}), \ldots, f_n(a_{i,n})) =$

$(b_{i,1}, \dots, b_{i,n})$, and hence $f(A) = B$. If $g(x) = (f_1^{-1}(x_1), \dots, f_n^{-1}(x_n))$, then g is the inverse of f. Therefore f is a homeomorphism. $\square$

Theorem 5.11. *Let $n \in \mathbb{Z}_{\geq 0}$. If a subset S of $\mathbb{R}^n$ satisfies* $\dim S = n$*, then S has an interior point.*

Proof. Suppose, toward a contradiction, that S has no interior point. Then $\mathbb{R}^n \setminus S$ is dense. We may therefore choose a countable dense subset A of $\mathbb{R}^n$ such that $A \cap S = \emptyset$. By Theorem 5.10, there exists a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(A) = \mathbb{Q}^n$. Since $\mathbb{R}^n \setminus \mathbb{Q}^n = \mathcal{M}(n, n-1)$, Lemma 2.12 gives $\dim(\mathcal{M}(n, n-1)) \leq n-1$. Hence $\dim f(S) \leq n - 1$. Since f is a homeomorphism, we conclude that $\dim S \leq n - 1$, contradicting the assumption $\dim S = n$. $\square$

6. The embedding theorem for finite-dimensional spaces

As an application of dimension theory, we prove in this section that every finite-dimensional separable metrizable space can be topologically embedded in a Euclidean space.

6.1. **Affine combinations and general position.**

Definition 6.1. Let $l \in \mathbb{Z}_{\geq 1}$. Finitely many points $e_1, \dots, e_m$ in the Euclidean space $\mathbb{R}^l$ are said to be **affinely independent** if, whenever $t_1, \dots, t_m \in \mathbb{R}$ satisfy $\sum_{i=1}^m t_i = 0$ and $\sum_{i=1}^m t_i \cdot e_i = 0$, we have $t_1 = \cdots = t_m = 0$. This is equivalent to the linear independence of $e_2 - e_1, \dots, e_m - e_1$.

An **affine combination** of finitely many points $w_1, \dots, w_m$ in $\mathbb{R}^l$ is a point of the form $\sum_{i=1}^m t_i \cdot w_i$, where $t_1, \dots, t_m \in \mathbb{R}$ and $\sum_{i=1}^m t_i = 1$. If $w_1, \dots, w_m$ are affinely independent, then the coefficients of each such affine combination are unique.

The **affine subspace** $\mathrm{Aff}(w_1, \dots, w_m)$ spanned by $w_1, \dots, w_m$ is the set of all their affine combinations. If W is the linear subspace spanned by $w_2 - w_1, \dots, w_m - w_1$, then $\mathrm{Aff}(w_1, \dots, w_m) = w_1 + W$. In particular, if $m \leq l$, then $\mathrm{Aff}(w_1, \dots, w_m)$ is a closed subset of $\mathbb{R}^l$ with empty interior.

Definition 6.2. Let $l \in \mathbb{Z}_{\geq 1}$. A subset $S \subset \mathbb{R}^l$ is said to be **in general position** if, for every $i \in \{1, \dots, l+1\}$, any i-many distinct elements of S are affinely independent.

Proposition 6.1. *Let $l \in \mathbb{Z}_{\geq 1}$, let $O \subset \mathbb{R}^l$ be a dense open set, and let $C \subset O$ be an at-most-countable set in general position. Then there exists a countable dense set $S \subset O$ such that $C \cup S$ is in general position and $C \cap S = \emptyset$.*

Proof. Choose a countable dense subset E of $\mathbb{R}^l$, and enumerate $E \times \mathbb{Q}_{>0}$ as $E \times \mathbb{Q}_{>0} = \{(q_k, r_k)\}_{k \in \mathbb{Z}_{\geq 1}}$. We inductively construct a sequence $e_1, \dots, e_k$ such that $C \cup \{e_1, \dots, e_k\}$ is in general position and we have $\|e_i - q_i\| < r_i$ for every $i \in \{1, \dots, k\}$.

Assume that $e_1, \dots, e_k$ have been constructed. We construct e_{k+1}. For every finite subset I of $C \cup \{e_1, \dots, e_k\}$ with $\mathrm{card}(I) \leq l$, define $A(I) = \mathrm{Aff}(I)$, and let A be the union of all such $A(I)$. Each $A(I)$ is closed and has empty interior. Since A is a countable union of such sets, the Baire category theorem implies that A also has empty interior. Thus $O \setminus A$ is dense. Since C is at most countable, $O \setminus (A \cup C)$ is also dense. We can therefore choose $e_{k+1} \in O \setminus (A \cup C)$ such that $\|e_{k+1} - q_{k+1}\| < r_{k+1}$. For every finite subset I of $C \cup \{e_1, \dots, e_k\}$ with $\mathrm{card}(I) \leq l$, we have $e_{k+1} \notin A(I)$. It follows that $C \cup \{e_1, \dots, e_k, e_{k+1}\}$ is in general position. □

Proposition 6.2. *Let $n, l \in \mathbb{Z}_{\geq 1}$, and $e_1, \dots, e_m \in \mathbb{R}^l$ be a sequence in general position in $\mathbb{R}^l$. If $2n + 1 \leq l$, then the coefficients of an affine combination $\sum_{i=1}^{m} s_i \cdot e_i$ whose support is a subset I of $\{1, \dots, m\}$ of cardinality at most $n + 1$ are unique. Here, to say that I is the support means that $s_i \neq 0$ if and only if $i \in I$.*

Proof. Assume that an affine combination $\sum_{i=1}^{m} t_i \cdot e_i$ with support J, where J is a subset of $\{1, \dots, m\}$ of cardinality at most $n + 1$, is equal to $\sum_{i=1}^{m} s_i \cdot e_i$. Then

$$\sum_{i \in I \setminus J} s_i \cdot e_i + \sum_{i \in J \setminus I} (-t_i) \cdot e_i + \sum_{i \in J \cap I} (s_i - t_i) \cdot e_i = 0. \tag{6.1}$$

Considering the coefficients in this equation gives

$$\sum_{i \in I \setminus J} s_i + \sum_{i \in J \setminus I} (-t_i) + \sum_{i \in J \cap I} (s_i - t_i) = 0. \tag{6.2}$$

The cardinality of $I \cup J$ is at most $2n + 2$. Since $2n + 1 \leq l$, we have $2n + 2 \leq l + 1$. The general-position assumption therefore implies that $\{e_i\}_{i \in I \cup J}$ is affinely independent. Hence $s_i = 0$ on $I \setminus J$, $t_i = 0$ on $J \setminus I$, and $s_i = t_i$ on $I \cap J$. Since s_i and t_i have supports I and J, respectively, we must have $I = J$. It then follows that $I = J = I \cap J$ and $s_i = t_i$ on I. This proves uniqueness of the coefficients. □

6.2. **Function spaces and embeddings.**

Definition 6.3. Let X be a topological space and (K, d) a compact metric space. In this manuscript, $\mathrm{C}(X, K)$ denotes the set of all continuous maps from X to K. The supremum metric on $\mathrm{C}(X, K)$ induced by d is denoted by $\widehat{d}$; thus $\widehat{d}(f, g) = \sup_{x \in X} d(f(x), g(x))$. By our definition of $\mathrm{C}(X, K)$, this metric is finite-valued.

The following standard proposition is stated without proof.

Proposition 6.3. *Let X be a topological space and (K, d) a compact metric space. Then $(\mathrm{C}(X, K), \widehat{d})$ is complete and, in particular, is a Baire space.*

Definition 6.4. Let $\mathcal{U}$ be a cover of a topological space X. A map $f \in \mathrm{C}(X, H)$ is called a $\mathcal{U}$-map (or $\mathcal{U}$-small) if, for every $z \in H$, there exist a neighborhood N of z and $U \in \mathcal{U}$ such that $f^{-1}(N) \subset U$. In this manuscript, $\mathrm{E}(\mathcal{U})$ denotes the set of all $\mathcal{U}$-maps in $\mathrm{C}(X, H)$.

Proposition 6.4. *Let X be a topological space, $\mathcal{U}$ a cover of X, and (K, d) a compact metric space. Then $\mathrm{E}(\mathcal{U})$ is open in $\mathrm{C}(X, K)$.*

Proof. Fix $f \in \mathrm{E}(\mathcal{U})$. For every $z \in K$, choose an open neighborhood N_z of z such that $f^{-1}(N_z) \subset U$ for some $U \in \mathcal{U}$. Since K is compact, the cover $\{N_z\}_{z \in K}$ has a Lebesgue number $r \in (0, \infty)$.

Let $g \in \mathrm{C}(X, K)$ satisfy $\widehat{d}(f, g) < 2^{-3}r$, and fix $z \in K$. Put $M = \mathbf{U}(z, 2^{-3}r; d)$. Choose $p(z) \in K$ such that $\mathbf{U}(z, r; d) \subset N_{p(z)}$. If $x \in g^{-1}(M)$, then $d(z, g(x)) \leq 2^{-3}r$. Moreover, $\widehat{d}(g, f) < 2^{-3}r$ implies $d(g(x), f(x)) < 2^{-3}r$. Hence $f(x) \in \mathbf{U}(z, r; d) \subset N_{p(z)}$. By the definition of $N_{p(z)}$, there exists $U \in \mathcal{U}$ such that $f^{-1}(N_{p(z)}) \subset U$. Thus $x \in U$. Since $x \in g^{-1}(M)$ was arbitrary, $g^{-1}(M) \subset U$, which completes the proof. $\square$

Proposition 6.5. *Let X be a metric space with $\dim X \leq n$, and let $\mathcal{U}$ be a finite cover of X. Assume that $l \in \mathbb{Z}_{\geq 1}$ satisfies $2n + 1 \leq l$, that K is a compact subset of $\mathbb{R}^l$ whose interior $\operatorname{Int} K$ is convex, and that $\mathrm{CL}(\operatorname{Int} K) = K$. Let d be the metric induced by the norm associated with the standard inner product on $\mathbb{R}^l$. Then $\mathrm{E}(\mathcal{U})$ is dense in $\mathrm{C}(X, K)$.*

Proof. Fix $f \in \mathrm{C}(X, K)$ and $\eta \in (0, \infty)$. Cover K by finitely many open sets $O_1, \dots, O_q$ of diameter at most $2^{-10}\eta$. Then the family $\{U \cap f^{-1}(O_i) \mid U \in \mathcal{U}, i = 1, \dots, q\}$ is a finite cover of X. Since $\dim X \leq n$, it has a finite refinement $\mathcal{V} = \{V_1, \dots, V_m\}$ such that $\deg(\mathcal{V}) \leq n + 1$. Choose functions $w_i \colon X \to [0, 1]$ whose cozero sets are V_i, and normalize them so that $\sum_{i=1}^m w_i = 1$. For example, one may begin with $w_i(x) = d(x, X \setminus V_i)$. Notice that the diameter of $f(V_i)$ is at most $2^{-10}\eta$.

Apply Proposition 6.1 with the dense subset $\mathbb{R}^l$ and $C = \emptyset$ to obtain a dense set $E \subset \mathbb{R}^l$ in general position. For $i \in \{1, \dots, m\}$, choose $e_i \in E \cap \operatorname{Int} K$ so that $d(f(V_i), e_i) < 2^{-5}\eta$. This is possible because $\mathrm{CL}(\operatorname{Int} K) = K$. Define $g \colon X \to K$ by $g(x) = \sum_{i=1}^m w_i(x) \cdot e_i$. For every $x \in X$, we have

$$\begin{aligned} d(f(x), g(x)) &= \left\| f(x) - \sum_{i=1}^{m} w_i(x) \cdot e_i \right\| \\ &= \left\| \sum_{i=1}^{m} w_i(x) \cdot f(x) - \sum_{i=1}^{m} w_i(x) \cdot e_i \right\| \\ &\leq \sum_{i=1}^{m} w_i(x) \cdot \| f(x) - e_i \| \end{aligned}$$

If $w_i(x) > 0$, then the estimates for the diameter of $f(V_i)$ and the choice of e_i show that the distance between $f(x)$ and e_i is at most $(2^{-5} + 2^{-10})\eta$. The displayed inequality therefore gives $\widehat{d}(f, g) < \eta$.

We next show that $g \in \mathrm{E}(\mathcal{U})$. Let R be the set of all convex combinations of at most $n+1$ members of $e_1, \ldots, e_m$. It is compact, since it is a finite union of simplices of dimension at most n, and the image of g is contained in R. If $z \in K \setminus R$, take N_z to be the complement of R. Then $g^{-1}(N_z) = \emptyset$, so the required condition holds.

Now let $z \in R$. Write $z = \sum_{i \in I} c_i \cdot e_i$, where $I \subset \{1, \ldots, m\}$, $\mathrm{card}(I) \leq n+1$, and $c_i \neq 0$. The definition of an affine combination implies $I \neq \emptyset$. By Proposition 6.2, we may choose sufficiently small $\epsilon \in (0, \infty)$ so that, whenever $y \in \mathbf{U}(z, \epsilon; d) \cap R$ is written as $y = \sum_i a_i \cdot e_i$, we have $a_i \neq 0$ for $i \in I$. If the image of g does not meet $\mathbf{U}(z, \epsilon; d)$, its inverse image is empty and the condition for a $\mathcal{U}$-map holds. Otherwise, put $N = \mathbf{U}(z, \epsilon; d)$. For $x \in X$ with $g(x) \in N$, choose $i \in I$. By the choice of ϵ, we have $w_i(x) > 0$, and hence $x \in V_i$. Thus $g^{-1}(N) \subset V_i$. This proves that $g \in \mathrm{E}(\mathcal{U})$, and consequently $\mathrm{E}(\mathcal{U})$ is dense in $\mathrm{C}(X, K)$. □

For a cover $\mathcal{U}$ of a topological space X and $x \in X$, define $\mathrm{St}(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid x \in U\}$.

A sequence $\mathcal{U}_0, \ldots, \mathcal{U}_n, \ldots$ of finite covers of a topological space X is called a **fundamental sequence of covers** if $\{\mathrm{St}(x, \mathcal{U}_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ is a neighborhood base at x for every $x \in X$.

Lemma 6.6. *Every separable metrizable space X admits a fundamental sequence of covers.*

Proof. By the Urysohn metrization theorem, X can be topologically embedded in $[0,1]^{\aleph_0}$. Thus there is a metric d inducing the topology of X such that (X, d) is totally bounded. Let $\mathcal{U}_n$ be a finite cover of X by open balls of radius at most 2^{-n}. Such a cover exists because (X, d) is totally bounded. Then $\{\mathcal{U}_n\}_{n \in \mathbb{Z}_{\geq 0}}$ is a fundamental sequence of covers. □

Proposition 6.7. *Let X be a metrizable space, (K, d) a compact metric space, and $\{\mathcal{U}_i\}_{i \in \mathbb{Z}_{\geq 0}}$ a fundamental sequence of covers. Then every member of $\bigcap_{i \in \mathbb{Z}_{\geq 0}} \mathrm{E}(\mathcal{U}_i)$ is a topological embedding.*

Proof. Fix $f \in \bigcap_{i \in \mathbb{Z}_{\geq 0}} \mathrm{E}(\mathcal{U}_i)$. We first prove injectivity. Let $x \neq y$. By the definition of a fundamental sequence, there exists $i \in \mathbb{Z}_{\geq 0}$ such that $y \notin \mathrm{St}(x, \mathcal{U}_i)$. Since $f \in \mathrm{E}(\mathcal{U}_i)$, there exist a neighborhood N of $f(x)$ and $U \in \mathcal{U}_i$ such that $f^{-1}(N) \subset U$. Now $y \notin \mathrm{St}(x, \mathcal{U}_i)$ implies $y \notin U$, and hence $f(y) \notin N$. In particular, $f(x) \neq f(y)$.

We next prove that f is a topological embedding. Fix $x \in X$ and a neighborhood V of x. There exists $i \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{St}(x, \mathcal{U}_i) \subset V$. Since $f \in \mathrm{E}(\mathcal{U}_i)$, there exist a neighborhood N of $f(x)$ and $U \in \mathcal{U}_i$

such that $f^{-1}(N) \subset U$. Therefore $f^{-1}(N) \subset \mathrm{St}(x, \mathcal{U}_i) \subset V$. This proves that f^{-1} is continuous at $f(x)$, and completes the proof. □

We write $\mathrm{Emb}(X, K)$ for the set of all topological embeddings from X into K.

Theorem 6.8. *Let X be a separable metric space with* $\mathrm{dim} X \leq n$. *Assume that $l \in \mathbb{Z}_{\geq 1}$ satisfies $2n + 1 \leq l$, and that K is a compact subset of $\mathbb{R}^l$ whose interior* $\mathrm{Int}\, K$ *is convex and satisfies* $\mathrm{CL}(\mathrm{Int}\, K) = K$. *Let d be the metric induced by the norm associated with the standard inner product on $\mathbb{R}^l$. Then the set* $\mathrm{Emb}(X, K)$ *of all topological embeddings from X into K is a dense G_δ subset of* $\mathrm{C}(X, K)$. *In particular, every finite-dimensional separable metrizable space can be embedded in a Euclidean space.*

Proof. Combine Propositions 6.3, 6.4, and 6.5 with Lemma 6.6. □

6.3. **Universal spaces.** We next construct a separable metric space into which every separable metrizable space of dimension at most n can be embedded.

Let $l \in \mathbb{Z}_{\geq 0}$, let X be a topological space, K a subset of $\mathbb{R}^l$, and M an affine subspace of $\mathbb{R}^l$. We write $\mathrm{Avoid}(M)$ for the set of all $f \in \mathrm{C}(X, K)$ such that $\mathrm{CL}(f(X)) \cap M = \emptyset$.

Proposition 6.9. *Let X be a metric space with* $\mathrm{dim} X \leq n$, *and let $\mathcal{U}$ be a finite cover of X. Assume that $l \in \mathbb{Z}_{\geq 1}$ satisfies $2n + 1 \leq l$, and that K is a compact subset of $\mathbb{R}^l$ whose interior* $\mathrm{Int}\, K$ *is convex and satisfies* $\mathrm{CL}(\mathrm{Int}\, K) = K$. *Then, for every affine subspace M of $\mathbb{R}^l$ of dimension at most n, the set* $\mathrm{Avoid}(M)$ *is dense and open.*

Proof. We first show that $\mathrm{Avoid}(M)$ is open. Fix $h \in \mathrm{Avoid}(M)$. Then $\mathrm{CL}(h(X)) \cap M = \emptyset$, and $\mathrm{CL}(h(X))$ is compact because K is compact. Hence, for sufficiently small η, the distance between $\mathrm{CL}(h(X))$ and M is at least η. If $\widehat{d}(f, h) < 2^{-1}\eta$, then $\mathrm{CL}(f(X)) \cap M = \emptyset$. Thus $\mathrm{Avoid}(M)$ is open.

We next prove density. Fix $f \in \mathrm{C}(X, K)$. Choose affinely independent points $c_1, \ldots, c_q$ spanning M. Notice that $q \leq n + 1$. Apply Proposition 6.1 to $C = \{c_1, \ldots, c_q\}$ to obtain a countable dense set $E \subset \mathbb{R}^l$ such that $C \cup E$ is in general position and $C \cap E = \emptyset$. Repeat the construction in the proof of Proposition 6.5, with this choice of E, to obtain $g(x) = \sum_{i=1}^{m} w_i(x) \cdot e_i$. The proof of Proposition 6.5 also shows that g approximates f.

We show that $g \in \mathrm{Avoid}(M)$. Let R be the set of all convex combinations of at most $n+1$ members of $e_1, \ldots, e_m$. It is compact, since it is a finite union of simplices of dimension at most n, and the image of g is contained in R. Proposition 6.2 and $E \cap C = \emptyset$ imply $R \cap M = \emptyset$. Consequently, $\mathrm{CL}(g(X)) \cap M = \emptyset$, and hence $g \in \mathrm{Avoid}(M)$. □

Fix $n \in \mathbb{Z}_{\geq 0}$. Let N_n^{2n+1} denote the subspace of $[0,1]^{2n+1}$ consisting of all points having at most n rational coordinates; that is, $\mathrm{N}_n^{2n+1} = [0,1]^{2n+1} \cap \mathcal{M}(2n+1, n)$. This space is also called the Nöbeling space. We prove that it is a universal space for n-dimensional separable metric spaces.

Theorem 6.10. *Let $n \in \mathbb{Z}_{\geq 0}$, and let X be a separable metrizable space with* $\mathrm{dim}X \leq n$*. Then there exists a topological embedding of X into* N_n^{2n+1}.

Proof. For a subset I of $\{1, \dots, 2n+1\}$ of cardinality $n+1$, say $I = \{u_1, \dots, u_{n+1}\}$, and $q = (q_1, \dots, q_{n+1}) \in \mathbb{Q}^{n+1}$, define

$$M(I; q) = \{x \in \mathbb{R}^{2n+1} \mid \forall i \in \{1, \dots, n+1\},\ x_{u_i} = q_i\} \tag{6.3}$$

Then $M(I; q)$ is an n-dimensional affine subspace. By the definition of N_n^{2n+1}, $[0,1]^{2n+1} \setminus \mathrm{N}_n^{2n+1} = [0,1]^{2n+1} \cap \bigcup_{I,q} M(I; q)$. Here I ranges over a finite set and q over a countable set.

Proposition 6.3, Theorem 6.8, and Proposition 6.7 show that the following set is dense and G_δ:

$$\mathrm{Emb}(X, [0,1]^{2n+1}) \cap \bigcap_{I,q} \mathrm{Avoid}(M(I; q)). \tag{6.4}$$

It follows that there exists a topological embedding of X into N_n^{2n+1}. □

Theorem 3.6 implies that the dimension of N_n^{2n+1} is n. We obtain the following.

Corollary 6.11. *Let X be a separable metrizable space with* $\mathrm{dim}X < \infty$*. Then X has a metrizable compactification Y such that* $\mathrm{dim}X = \mathrm{dim}Y$.

Proof. Put $n = \mathrm{dim}X$. Choose a map f from the set in (6.4). Then the closure Y of the image of f is contained in N_n^{2n+1}. Moreover, Y is compact because it is a closed subset of the compact space $[0,1]^{2n+1}$, and it satisfies $\mathrm{dim}X \leq \mathrm{dim}Y \leq n = \mathrm{dim}X$. This completes the proof. □

Remark 6.1. The book [2] uses Corollary 6.11 to prove the equality of the three dimensions on separable metrizable spaces (Theorem 2.29).

7. Embedding manifolds into Euclidean spaces

As another application of dimension theory, we prove in this section that every differentiable manifold can be embedded in a Euclidean space by a differentiable map.

7.1. **Limit sets.**

Definition 7.1. Let X be a second-countable, locally compact metrizable space. A sequence $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is said to **tend to infinity**, written $x_i \to \infty$, if, for every compact subset K, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $x_n \notin K$ for every $n \in \mathbb{Z}_{\geq 0}$ with $N \leq n$. Equivalently, x_i converges to the point at infinity ∞ in the one-point compactification of X.

Definition 7.2. For two second-countable, locally compact metrizable spaces X and Y and a continuous map $f\colon X \to Y$, define $L(f)$ by the set of all $y \in Y$ such that there exists $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}} \subset X$ such that $x_i \to \infty$ and $f(x_i) \to y$. This is called the **limit set** of f.

Proposition 7.1. *Every second-countable, locally compact Hausdorff space X admits a sequence $\{K_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of compact subsets satisfying*

$$X = \bigcup_{i\in\mathbb{Z}_{\geq 0}} K_i; \tag{7.1}$$

$$K_i \subset \operatorname{Int}_X(K_{i+1}) \tag{7.2}$$

Moreover, for every compact subset K of X, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $K \subset K_m$. This corresponds to choosing a countable neighborhood base at the point at infinity in the one-point compactification.

Remark 7.1. A space admitting a sequence $\{K_i\}_{i\in\mathbb{Z}_{\geq 0}}$ with the properties in the preceding proposition is called **hemicompact**. In general, every σ-compact locally compact Hausdorff space is hemicompact.

The final assertion of Proposition 7.1 immediately gives the following.

Proposition 7.2. *Let $\{K_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence of compact subsets of X as in Proposition 7.1. Then $x_i \to \infty$ if and only if, for every $m \in \mathbb{Z}_{\geq 0}$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $x_i \notin K_m$ for every $i \in \mathbb{Z}_{\geq 0}$ with $N \leq i$.*

We first record the following elementary fact.

Proposition 7.3. *Let X and Y be second-countable, locally compact metrizable spaces, and let $f\colon X \to Y$ be continuous. Then $L(f)$ is closed in Y.*

Proof. Choose a sequence $\{K_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of compact sets as in Proposition 7.1, and let e be a metric compatible with the topology of Y. Assume that $y_i \in L(f)$ and $y_i \to y \in Y$. After passing to a subsequence, we may assume without loss of generality that $e(y_i, y) < 2^{-i}$. For each $m \in \mathbb{Z}_{\geq 0}$, there exists a sequence $\{x_i^{(m)}\}_{i\in\mathbb{Z}_{\geq 0}}$ in X such that $x_i^{(m)} \to \infty$ and $\lim_{i\to\infty} f(x_i^{(m)}) = y_m$.

Choose a strictly increasing map $\phi\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ so that, whenever $i \geq \phi(m)$, we have $x_i^{(m)} \notin K_m$ and $e(y_m, f(x_i^{(m)})) < 2^{-m}$. Put $z_i = x_{\phi(i)}^{(i)}$. Then $z_i \notin K_i$ for every i. Since $K_i \subset K_{i+1}$, if $n \geq N$, then $z_n \notin$

K_N. Proposition 7.2 therefore gives $z_i \to \infty$. Moreover, $e(y, f(z_i)) \leq e(y, y_i) + e(y_i, f(z_i)) \leq 2^{-i+1}$. Thus $f(z_i) \to y$, so $y \in L(f)$. Hence $L(f)$ is closed. □

Proposition 7.4. *Let X and Y be second-countable, locally compact metrizable spaces, and let $f\colon X \to Y$ be a continuous injection. Then f is a topological embedding if and only if $L(f) \cap f(X) = \emptyset$.*

Proof. First assume that f is a topological embedding. Suppose, toward a contradiction, that $L(f) \cap f(X) \neq \emptyset$. Choose $y \in L(f) \cap f(X)$. Then there exist a sequence $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in X and a point $x \in X$ such that $x_i \to \infty$, $f(x_i) \to y$, and $y = f(x)$. Since $x_i \to \infty$, the sequence does not converge to any point of X. Thus, although $f(x_i) \to y$,

$$\lim_{i\to\infty} f^{-1}(f(x_i)) \neq f^{-1}(y)(= x). \tag{7.3}$$

Hence $f^{-1}\colon f(X) \to X$ is not continuous, contradicting the assumption that f is an embedding. Therefore $L(f) \cap f(X) = \emptyset$.

Conversely, assume that $L(f) \cap f(X) = \emptyset$. Now we prove that $f^{-1}\colon f(X) \to X$ is continuous. It is enough to prove that, for every $A \subset f(X)$,

$$f^{-1}(\mathrm{CL}_{f(X)}(A)) \subset \mathrm{CL}_X(f^{-1}(A)). \tag{7.4}$$

Let $y \in \mathrm{CL}_{f(X)}(A)$. There exists a sequence $\{y_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in A converging to y. Put $x_i = f^{-1}(y_i)$. Since $y \in f(X)$ and $L(f) \cap f(X) = \emptyset$, we have $y \notin L(f)$, and hence $x_i \not\to \infty$. Thus $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ has an accumulation point and consequently a convergent subsequence $\{x_{\phi(i)}\}_{i\in\mathbb{Z}_{\geq 0}}$. Let $x \in X$ be its limit. By continuity of f, we have $f(x_{\phi(i)}) \to f(x)$. Since $f(x_{\phi(i)}) = y_{\phi(i)}$ and every subsequence of a convergent sequence converges to the same limit, we obtain $f(x) = y$, and hence $x = f^{-1}(y)$. Thus, after passing to a subsequence, $f^{-1}(y_{\phi(i)}) \to f^{-1}(y)$. Since this is a sequence in $f^{-1}(A)$, the required inclusion follows. Therefore f^{-1} is continuous. □

Proposition 7.5. *Let X and Y be second-countable, locally compact metrizable spaces, and let $f\colon X \to Y$ be continuous. Then $f(X)$ is closed in Y if and only if $L(f) \subset f(X)$.*

Proof. If $f(X)$ is closed in Y, then $L(f) \subset f(X)$ follows immediately from the definition of the limit set.

Conversely, assume that $L(f) \subset f(X)$. Let $\{y_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence in $f(X)$ such that $y_i \to y$, and choose $x_i \in f^{-1}(y_i)$. If $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ has a convergent subsequence, continuity of f implies that its limit x satisfies $f(x) = y$, so $y \in f(X)$. If it has no convergent subsequence, that is, if $x_i \to \infty$, then $y \in L(f) \subset f(X)$. In either case, $y \in f(X)$, and hence $f(X)$ is closed in Y. □

We consequently obtain the following.

Corollary 7.6. *Let X and Y be second-countable, locally compact metrizable spaces. Assume that $f\colon X \to Y$ is injective. If $L(f) = \emptyset$, then f is a closed map and a topological embedding.*

7.2. **Constructing an embedding.** For two C^r differentiable manifolds M and N, let $\mathrm{Emb}^r(M, N)$ denote the set of all C^r embeddings from M to N. We now prove the existence of embeddings of manifolds into Euclidean spaces.

Theorem 7.7. *Let N be an n-dimensional C^r manifold. Then we have $\mathrm{Emb}^r(N, \mathbb{R}^{(n+1)^2}) \neq \emptyset$.*

Proof. By Proposition 2.14, $\mathbb{D}\mathrm{im} N \leq n$. In fact equality holds, but we do not need this sharper estimate. Let $\mathcal{A}$ be a cover of N by relatively compact coordinate neighborhoods. Applying Theorem 2.21 to $\mathcal{A}$, we obtain a refinement $\mathcal{B}$ and $n+1$ families of open sets $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$, not necessarily themselves covers, such that each $\mathcal{U}_i$ is a pairwise disjoint family and

$$\mathcal{B} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

Using normality of N, we may further assume that $\mathrm{CL}(V) \cap \mathrm{CL}(W) = \emptyset$ whenever $V, W \in \mathcal{U}_i$ are distinct. Enumerate $\mathcal{U}_k = \{U_i^{(k)}\}_{i \in \mathbb{Z}_{\geq 0}}$.

By the shrinking lemma for normal spaces, Lemma 1.5, choose finite sequences of families $\{\mathcal{V}_k\}_{k=0}^n$ and $\{\mathcal{W}_k\}_{k=0}^n$, where $\mathcal{V}_k = \{V_i^{(k)}\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\mathcal{W}_k = \{W_i^{(k)}\}_{i \in \mathbb{Z}_{\geq 0}}$, such that $\mathrm{CL}(V_i^{(k)}) \subset U_i^{(k)}$, $\mathrm{CL}(W_i^{(k)}) \subset V_i^{(k)}$, and both $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_n$ and $\mathcal{W}_0 \cup \cdots \cup \mathcal{W}_n$ cover N.

Choose $A_i^{(k)} \in \mathcal{A}$ so that $\mathrm{CL}(V_i^{(k)}) \subset A_i^{(k)}$. There are coordinate charts $\phi_i^{(k)}$ on $A_i^{(k)}$ with the following properties:

(P1) If $i \neq j$, then $\phi_i^{(k)}(\mathrm{CL}(V_i^{(k)})) \cap \phi_j^{(k)}(\mathrm{CL}(V_j^{(k)})) = \emptyset$.

(P2) If a sequence $\{x_s\}_{s \in \mathbb{Z}_{\geq 0}}$ in N satisfies $x_s \in V_{i_s}^{(k)}$ for every $s \in \mathbb{Z}_{\geq 0}$ and $x_s \to \infty$, then $\phi_{i_s}^{(k)}(x_s) \to \infty$.

Their existence follows by composing with translations in Euclidean space and using the fact that a compact set meets only finitely many members of a locally finite family.

For each i and k, let $b_i^{(k)}\colon N \to \mathbb{R}$ be a nonnegative C^r function equal to 1 on $V_i^{(k)}$ and to 0 outside $U_i^{(k)}$. Define $\theta^{(k)}\colon N \to \mathbb{R}^n$ by

$$\theta^{(k)}(x) = \begin{cases} b_i^{(k)}(x) \cdot \phi_i^{(k)}(x) & x \in U_i^{(k)}; \\ 0 \in \mathbb{R}^n & x \notin \bigcup_i U_i^{(k)}. \end{cases}$$

Notice that $\theta^{(k)}|V_i^{(k)} = \phi_i^{(k)}$.

Since $\mathrm{CL}(W_i^{(k)}) \subset V_i^{(k)}$,

$$\phi_i^{(k)}\left(\mathrm{CL}\left(W_i^{(k)}\right)\right) \subset \phi_i^{(k)}\left(V_i^{(k)}\right). \tag{7.5}$$

Choose a nonnegative smooth function $p_i^{(k)}\colon \mathbb{R}^n \to \mathbb{R}$ equal to 1 on $\phi_i^{(k)}(W_i^{(k)})$ and to 0 outside $\phi_i^{(k)}(V_i^{(k)})$. Define $\eta^{(k)}\colon N \to \mathbb{R}$ by

$$\eta^{(k)}(x) = \begin{cases} p_i^{(k)} \circ \phi_i^{(k)}(x) & x \in V_i^{(k)}; \\ 0 \in \mathbb{R} & x \notin \bigcup_i V_i^{(k)}. \end{cases} \tag{7.6}$$

Then $\eta^{(k)}$ is of class C^r.

Put $l = (n+1)^2$ and define $\Phi\colon N \to \mathbb{R}^l$ by

$$\Phi(x) = (\theta^{(0)}(x), \dots, \theta^{(n)}(x), \eta^{(0)}(x), \dots, \eta^{(n)}(x)). \tag{7.7}$$

We prove that Φ is a differentiable embedding.

First, the differential of Φ has full rank. This follows from the fact that the sets $W_i^{(k)}$ cover N and from the definition of $\theta^{(k)}$.

We next prove injectivity. Assume that $\Phi(x) = \Phi(y)$. Since the sets $W_i^{(k)}$ cover N, there exists $k \in \{0, \dots, n\}$ such that $0 < \eta^{(k)}(x)(= \eta^{(k)}(y))$. Thus $x, y \in \bigcup_{i \in \mathbb{Z}_{\geq 0}} V_i^{(k)}$. Choose $i(x), i(y) \in \mathbb{Z}_{\geq 0}$ such that $x \in V_{i(x)}^{(k)}$ and $y \in V_{i(y)}^{(k)}$. The equality $\Phi(x) = \Phi(y)$ also gives $\theta^{(k)}(x) = \theta^{(k)}(y)$. By the definition of $b_i^{(k)}$ and the fact that $b_i^{(k)}$ is 1 on $V_i^{(k)}$, we see that $\theta^{(k)}(x) = \phi_{i(x)}^{(k)}(x)$ and $\theta^{(k)}(y) = \phi_{i(y)}^{(k)}(y)$. By (P1) and $\theta^{(k)}(x) = \theta^{(k)}(y)$, we obtain $i(x) = i(y) = i_0$. Since $\phi_{i_0}^{(k)}$ is a coordinate chart, we conclude that $x = y$.

It remains to prove that Φ is a topological embedding. Let $L(\Phi)$ be its limit set. By Proposition 7.4, it is enough to prove that $L(\Phi) = \emptyset$. Let $\{x_s\}_{s \in \mathbb{Z}_{\geq 0}}$ be a sequence in N such that $x_s \to \infty$. After passing to a subsequence, we may assume without loss of generality that, for some $k \in \{0, \dots, n\}$, we have

$$x_s \in \bigcup_{i \in \mathbb{Z}_{\geq 0}} V_i^{(k)}. \tag{7.8}$$

By the choice of the coordinate charts $\phi_i^{(k)}$, we have $\theta^{(k)}(x_s) \to \infty$. In particular, $\Phi(x_s) \to \infty$. This means that $L(\Phi) = \emptyset$, and completes the proof. □

Remark 7.2. The proof also shows that Φ is a closed map. An application of Sard's theorem gives $\mathrm{Emb}^r(N, \mathbb{R}^{2n+1}) \neq \emptyset$. Using techniques from differential topology, one can moreover prove that, if $2n + 1 \leq l$, then $\mathrm{Emb}^r(N, \mathbb{R}^l)$ is dense in the space of differentiable maps.

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