A MEMO ON PARACOMPACTNESS

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Abstract. On a paracompact Hausdorff space, every open coverings has a refinement that is a σ-discrete and locally finite open covering.

1. Explanation

Theorem 1.1. *Let X be a paracompact Hausdorff space and $\mathfrak{U}$ be its open covering. Then there exists a σ-discrete and locally finite open covering of X that refines $\mathfrak{U}$.*

Proof. Take families $\{U_{a,i}\}_{a\in A, i\in\mathbb{Z}_{\geq 0}}$ of open sets and $\{F_{a,i}\}_{a\in A, i\in\mathbb{Z}_{\geq 0}}$ closed subsets such that

(1) For a fixed integer $n \in \mathbb{Z}_{\geq 0}$, the families $\{U_{a,n}\}_{a\in A}$ and $\{F_{a,n}\}_{a\in A}$ are locally finite families of X that refine $\mathfrak{U}$;
(2) for each $a \in A$, and $i \in \mathbb{Z}_{\geq 0}$ we have $U_{a,i} \subset F_{a,i} \subset U_{a,i+1}$;
(3) Put $U_{a,\infty} = \bigcup_{n\in\mathbb{Z}_{\geq 0}} U_{a,n}$, then the family $\{U_{a,\infty}\}_{a\in A}$ is an open covering refining $\mathfrak{U}$.

To guarantee the existence of two coverings stated above, for example, for a locally finite partition of unity $\{f_a\}_{a\in A}$, put $F_{a,i} = f^{-1}([2^{-i}, 1])$ and $U_{a,i} = f^{-1}((2^{-i}, 1])$.

Equip A with a well-ordering $<$. We define $E_{a,i}$ by

$$E_{a,i} = U_{a,i} \setminus \bigcup_{p<a} F_{p,i+1}. \tag{1}$$

By the local finiteness of $\{F_{a,i+1}\}_{a\in A}$, the set $E_{a,i}$ is open. Now we prove $\{E_{a,i}\}_{a\in A, i\in\mathbb{Z}_{\geq 0}}$ is a covering. Take $x \in X$ and take minimum $a \in A$ with $x \in U_{a,\infty}$. Then there exists $i \in \mathbb{Z}_{\geq 0}$ with $a \in U_{a,i}$. By the minimality of a, for every $p < a$ satisfies that $x \notin U_{p,\infty}$. Since $F_{p,i+1} \subset U_{p,\infty}$, we also have $x \notin F_{p,i+1}$. Thus $\{E_{a,i}\}_{a\in A, i\in\mathbb{Z}_{\geq 0}}$ is a covering. By the definition, it also refines $\mathfrak{U}$.

Next, fix $n \in \mathbb{Z}_{\geq 0}$, we show that $\{E_{a,n}\}_{a\in A}$ is a discrete family. Take $x \in X$ and the minimum $a \in A$ with $x \in U_{a,\infty}$. We also take $i \in \mathbb{Z}_{\geq 0}$ with $x \in F_{a,i}$. Then $E_{a,i+1}$ is a neighborhood of x and for all $a < s$ $E_{a,i+1} \cap E_{s,i} = \emptyset$ since $E_{a,i+1} \subset F_{a,i+1}$. If $s < a$, then $E_{a,i+1} \cap E_{s,i} = \emptyset$ by the minimality of $a \in A$. Thus, $E_{a,i+1}$ meets only $E_{a,i}$. This means that the family is discrete.

Date: June 15, 2026.
Key words and phrases. Paracompactness.

Define $O_n = \bigcup_{a \in A} E_{a,n}$. Then $\{O_n\}_{n \in \mathbb{Z}_{\geq 0}}$ is an open covering of X, and hence we can take a locally finite open covering $\{P_n\}_{n \in \mathbb{Z}_{\geq 0}}$ such that $P_n \subset O_n$. Put $\mathcal{V} = \{\, P_n \cap E_{a,n} \mid a \in A, n \in \mathbb{Z}_{\geq 0} \,\}$. Therefore $\mathcal{V}$ is a σ-discrete and locally finite open covering of X that refines $\mathcal{U}$. $\square$

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