MEMO: DESCRIBING PERFECT MAPS WITH PULLBACKS

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In this note, following [1], we describe a characterization of perfect maps by means of pullbacks. For the parts I did not understand very well, I also referred to [3].

The English in this manuscript was written with the assistance of AI tools, including Codex.

1. Pullbacks

I assume that the readers already know commutative diagrams.

Let X, Y, Z be topological spaces, and consider maps $f\colon X \to Z$ and $g\colon X \to Z$.

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

In this situation, a topological space P together with two maps $g'\colon P \to X$ and $f'\colon P \to Y$ is called a **pullback** of the above diagram if the following diagram is commutative,

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

and, whenever another T, $b\colon T \to X$, and $a\colon T \to Y$ making the diagram commute are given, there exists a unique u making the following diagram commute.

$$\begin{tikzcd} T \arrow[dr, dashed, "u"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\ & P \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

To indicate that P is a pullback, we write a right-angle symbol as follows.

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$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

The map f' is sometimes called a base change. The pullback P is sometimes denoted by $X \times_{Z,f,g} Y$ or $X \times_Z Y$.

From the above definition, we see that a pullback, if it exists, is unique up to isomorphism. We now check several basic facts about pullbacks of topological spaces.

Theorem 1. *In the category of topological spaces, pullbacks always exist. In other words, in the situation above, the pullback P always exists. Moreover, as a concrete description of the pullback, we may take $P = \{\, (x, y) \in X \times Y \mid f(x) = g(y) \,\}$, with f' and g' the projections.*

Proof. Since we have given a concrete description, it is enough to check commutativity and the universal property. If you move your hands and compute briskly, it becomes clear. $\square$

Although pullbacks of topological spaces can be written concretely as above, what seems important is to know what they can be identified with in special situations.

Proposition 2. *The following statements hold.*

(A1) *In the following diagram, $f^{-1}(x)$ is a pullback. That is, for a map $f\colon X \to Z$, when one considers a point $y\colon \{*\} \to Z$, the base change of f is the generalized fiber $f^{-1}(y)$.*

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \{*\} \arrow[r, "y"] & Y \end{tikzcd}$$

(A2) *For a continuous map $f\colon X \to Y$ and a topological space S, the following diagram is a pullback.*

$$\begin{tikzcd} X \times S \arrow[r] \arrow[d, "f \times 1_S"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \times S \arrow[r, "pr"] & Y \end{tikzcd}$$

For a continuous map $f\colon X \to Y$ and a subset B of Y, the following diagram is a pullback. Here $\iota\colon B \to Y$ is the inclusion map.

(A3)

$$\begin{tikzcd} f^{-1}(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ B \arrow[r, "\iota"] & Y \end{tikzcd}$$

(A4) *For a continuous map $f\colon X \to Y$, the following trivial pullback diagram holds.*

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "1_Y"] & Y \end{tikzcd}$$

(A5) *Suppose that the following diagram is a pullback.*

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "v"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then for any topological space S, the following diagram is also a pullback.

$$\begin{tikzcd} P \times S \arrow[r, "h\times 1_S"] \arrow[d, "v\times 1_S"] \arrow[dr, phantom, "\lrcorner", very near start] & X \times S \arrow[d, "f\times 1_S"] \\ Y \times S \arrow[r, "g\times 1_S"] & Z \times S \end{tikzcd}$$

Proof. Proof of (A1): In the above situation, we have

$$X \times_Z \{*\} = \{\, (x, *) \in X \times \{*\} \mid f(x) = y \,\}. \tag{1.1}$$

Thus $X \times_Z \{*\}$ and $f^{-1}(y)$ are homeomorphic to each other.

Proof of (A2): This follows by briskly computing the universal property.

Proof of (A3): We have

$$P = X \times_Y B = \{\, (x, b) \in X \times B \mid f(x) = \iota(b) \,\}. \tag{1.2}$$

Define $\alpha\colon f^{-1}(B) \to P$ by $\alpha(x) = (x, f(x))$, and define $\beta\colon P \to f^{-1}(B)$ by $\beta(x, b) = x$. These maps are inverse to each other, hence P and $f^{-1}(B)$ are homeomorphic.

Proof of (A4): This follows by computing the universal property.

Proof of (A5): This follows by briskly computing the universal property.

In general, pullbacks commute with limits. In particular, pullbacks commute with products, so the product of two pullback diagrams is again a pullback.

Let us end the proof around here. □

Connecting pullbacks with each other again gives a pullback.

Theorem 3. *Consider the situation of the following commutative diagram.*

$$\begin{tikzcd}
P \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[r, "h'"] \arrow[d, "f=v'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "v"] \\
Y \arrow[r, "g"] & Z \arrow[r, "h"] & B
\end{tikzcd}$$

That is, in the diagram on the right, assume that X is the pullback of A and Z over B. Then P is the pullback of the diagram on the left if and only if P is the pullback of the large outer diagram, namely if and only if

$$\begin{tikzcd}
P \arrow[r, "h'\circ g'"] \arrow[d, "f'"] & A \arrow[d, "v"] \\
Y \arrow[r, "h\circ g"] & B
\end{tikzcd}$$

is a pullback diagram.

Proof. First suppose that P is the pullback of the small diagram. To prove the universal property of the large diagram, assume that continuous maps $p\colon T \to A$ and $q\colon T \to Y$ from a topological space T make the diagram commute.

$$\begin{tikzcd}
T \arrow[drrr, bend left, "p"] \arrow[dr, "q"'] & & & \\
& P \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "h'"] \arrow[d, "f=v'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "v"] \\
& Y \arrow[r, "g"] & Z \arrow[r, "h"] & B
\end{tikzcd}$$

Then $p\colon T \to A$ and $g \circ q\colon T \to Z$ make the diagram consisting of $v\colon A \to B$ and $h\colon Z \to B$ commute.

$$\begin{tikzcd}
T \arrow[drrr, bend left, "p"] \arrow[ddr, bend right, "q"'] \arrow[ddrr, bend right, "g\circ q"] & & & \\
& P \arrow[r, "g'"'] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "h'"] \arrow[d, "f=v'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "v"] \\
& Y \arrow[r, "g"'] & Z \arrow[r, "h"] & B
\end{tikzcd}$$

Since X is a pullback, there exists a unique map $r\colon T \to X$ making the diagram commute.

$$\begin{tikzcd}
T \arrow[drrr, bend left, "p"] \arrow[drr, "r"] \arrow[ddr, "q"'] & & & \\
& P \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "h'"] \arrow[d, "f=v'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "v"] \\
& Y \arrow[r, "g"] & Z \arrow[r, "h"] & B
\end{tikzcd}$$

Applying the universal property of P to $r\colon T \to X$ and $q\colon T \to Y$, we obtain a unique map $l\colon T \to P$.

$$\begin{tikzcd}
T \arrow[drrr, bend left, "p"] \arrow[drr, bend left, "r"] \arrow[dr, "l"] \arrow[ddr, "q"'] & & & \\
& P \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, "h'"] \arrow[d, "f=v'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "v"] \\
& Y \arrow[r, "g"] & Z \arrow[r, "h"] & B
\end{tikzcd}$$

Since all of the above diagrams commute, the following diagram also commutes.

$$\begin{tikzcd}
T \arrow[drr, bend left, "p"] \arrow[dr, dashed, "l"] \arrow[ddr, "q"'] & & \\
& P \arrow[r, "h'\circ g'"] \arrow[d, "f'"] & A \arrow[d, "v"] \\
& Y \arrow[r, "h\circ g"] & B
\end{tikzcd}$$

Uniqueness also follows from the uniqueness of r and l (why?). Thus P is $A \times_B Y$. We omit the converse. If one retraces the above operation in reverse, it should work. $\square$

By turning the diagram vertically, we also obtain the following.

Theorem 4. *Consider the situation of the following commutative diagram.*

$$\begin{tikzcd}
P \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y \arrow[r, "g=h'"] \arrow[d, "v'"] \arrow[r, phantom, "\lrcorner", near start, shift right=3] & Z \arrow[d, "v"] \\
A \arrow[r, "h"] & B
\end{tikzcd}$$

Then P is the pullback of the small square if and only if it is the pullback of the large outer diagram.

$$\begin{tikzcd}
P \arrow[r, "g'"] \arrow[d, "v' \circ f'"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "v \circ f"] \\
Y \arrow[r, "h"] & B
\end{tikzcd}$$

2. Perfect Maps

First, we explain that the base change of an embedding is again an embedding. In what follows, the word "an embedding" means "a topological embedding".

Proposition 5. *Consider the following pullback diagram.*

$$\begin{tikzcd}
P \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

If f is an embedding, then f' is also an embedding. Furthermore, if f is a closed embedding, then f' is also a closed embedding.

Proof. Since X is embedded, we may regard it as a subset of Z. By (A3) of Proposition 2, the pullback P is $g^{-1}(X)$. That is, we have

$$\begin{tikzcd}
g^{-1}(X) \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Here f' is the inclusion map into Y. In particular, it is an embedding. Also, when f is a closed embedding, the space X is a closed subset of Z, thus $g^{-1}(X)$ is a closed subset of Y. Hence f' is a closed embedding. This completes the proof. $\square$

Proposition 6. *Let X, Y be topological spaces, and let $f\colon X \to Y$ be a continuous map. Then the following hold.*

(T1) *X is Hausdorff if and only if the diagonal map $\Delta_X\colon X \to X \times X$ is a closed embedding.*

(T2) *If Y is Hausdorff, then the map $\Gamma_f \colon X \to X \times Y; x \mapsto (x, f(x))$ to the graph is a closed embedding.*

Proof. Proof of (T1): This follows immediately from the equivalence between the diagonal being closed and X being T_2.

Proof of (T2): This follows from the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & X \times Y \arrow[r, "pr"] \arrow[d, "f \times 1_Y"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "\Delta_X"] & Y \times Y \arrow[r, "pr"] & Y
\end{tikzcd}$$

Here note that, by (A4) of Proposition 2, the outer square is a pullback, and by (A2) of the same proposition, the diagram on the right is a pullback. Then by Theorem 3, the square on the left is a pullback. At this point, using Proposition 5 and the fact that Δ_X is a closed embedding, we see that its base change Γ_f is also a closed embedding. Done. □

Definition 7. *A continuous map $f \colon X \to Y$ is called a* ***perfect map*** *if f is a closed map and, for every $y \in Y$, the set $f^{-1}(y)$ is compact.*

Proposition 8. *The following hold.*

(K1) *If X is compact and Y is Hausdorff, then every continuous map $f \colon X \to Y$ is a perfect map.*

(K2) *A space K is compact if and only if the map $K \to \{*\}$ to a point is a perfect map.*

(K3) *A space K is compact if and only if, for every topological space S, the projection $pr \colon S \times K \to S$ is a closed map.*

Proof. (K1) and (K2) follow from compact Hausdorff-ness.

Using the tube lemma(see Lemma 16 in Appendix), one obtains the assertion in (K3) that the projection is closed. Next, to complete the proof of (K3), for a non-compact space X, we construct a space Y such that the projection $\pi \colon X \times Y \to Y$ is not a closed map.

Since X is non-compact, there exists an open cover $\mathcal{C}$ of X which has no finite subcover. For this $\mathcal{C}$, define the cardinal κ by

$$\kappa = \min\{\mathrm{card}(\mathcal{U}) \mid \mathcal{U} \subset \mathcal{C} \text{ and } \mathcal{U} \text{ is a cover of } X\}. \tag{2.1}$$

Here $\mathrm{card}(A)$ denotes the cardinality of the set A. By the choice of $\mathcal{C}$, the cardinal κ is infinite. By the minimality of κ, we can take a cover $\mathcal{U}$ such that $\kappa = \mathrm{card}(\mathcal{U})$. Indexing this cover by κ, write $\mathcal{U} = \{U_\alpha\}_{\alpha<\kappa}$. Now put $Y = [0, \kappa] = \kappa+1$, and define a family of open subsets $\{V_\alpha\}_{\alpha<\kappa}$ of Y by $V_\alpha = (\alpha, \to) = (\alpha, \kappa]$. We also define a subset O of $X \times Y$ by

$$O = \bigcup_{\alpha<\kappa} U_\alpha \times V_\alpha.$$

By definition, for each $\alpha < \kappa$, we have $\kappa \in V_\alpha$. Since $\{U_\alpha\}_{\alpha<\kappa}$ is a cover, for every $x \in X$ we have $(x, \kappa) \in O$. Moreover, O is clearly an open subset of $X \times Y$, so

$$F = (X \times Y) \setminus O \tag{2.2}$$

is a closed subset of $X \times Y$. We also see that for every $x \in X$, we have $(x, \kappa) \notin F$. Thus the image under the projection $\pi\colon X \times Y \to Y$ satisfies $\kappa \notin \pi(F)$. Now let us show that $\pi(F) = Y \setminus \{\kappa\}$. Let $\beta \in \kappa$ be arbitrary. Since $\mathrm{card}(\{U_\alpha\}_{\alpha<\beta}) = \mathrm{card}([0, \beta)) < \kappa$, by the minimality of κ, the family $\{U_\alpha\}_{\alpha<\beta}$ is not a cover of X. Thus we can take $x \in X \setminus \bigcup_{\alpha<\beta} U_\alpha$. Then

$$(x, \beta) \notin \bigcup_{\alpha<\beta} U_\alpha \times V_\alpha. \tag{2.3}$$

On the other hand, if $\beta \leq \alpha$, then $\beta \notin V_\alpha = (\alpha, \kappa]$, so

$$(x, \beta) \notin \bigcup_{\beta\leq\alpha<\kappa} U_\alpha \times V_\alpha. \tag{2.4}$$

Therefore $(x, \beta) \notin O$, and hence $(x, \beta) \in F$. This means that $\beta \in \pi(F)$. Consequently, we obtain $\pi(F) = Y \setminus \{\kappa\}$.

Since κ is an infinite cardinal, it is a limit ordinal. Thus, in $Y = [0, \kappa]$, the point κ is an accumulation point; in particular, $\{\kappa\}$ is not open. Therefore $\pi(F) = Y \setminus \{\kappa\}$ is not closed. Consequently, the projection $\pi\colon X \times Y \to Y$ is not a closed map. This finishes the proof of (K3). □

Remark. *The statement* (K3) *is known as Kuratowski-Mrówka's theorem.*

We now describe properties of perfect maps by using pullbacks.

Theorem 9. *Let X and Y be topological spaces, and let $f\colon X \to Y$ be a continuous map. Then the following are equivalent.*

- (P1) *the map f is perfect.*
- (P2) *For every space Z, the map $f \times 1_Z\colon X \times Z \to Y \times Z$ is perfect.*
- (P3) *For every $g\colon Y' \to Y$, the base change $f'\colon X' \to X$ is a perfect map.*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

- (P4) *For every space Z, the map $f \times 1_Z\colon X \times Z \to Y \times Z$ is closed.*

(P5) *For every $g\colon Y' \to Y$, the base change $f'\colon X' \to X$ is a closed map.*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Proof. We prove the implications as follows.

$$\begin{tikzcd} & & (P5) \arrow[dr, Rightarrow] & & \\ (P1) \arrow[r, Rightarrow] & (P3) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & (P2) \arrow[r, Rightarrow] & (P4) \arrow[r, Rightarrow] & (P1) \end{tikzcd}$$

Proof of (P1) $\Longrightarrow$ (P3): First we show that the fibers are compact. Consider the following diagram. Take an arbitrary $p \in Y'$, and put $q = g(p) \in Y$. The out ermost diagram is a pullback by (A1) of Proposition 2, and the diagram on the right is a pullback by the assumption (P1).

$$\begin{tikzcd} f^{-1}(q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \{*\} \arrow[r, hook, "p"] \arrow[rr, bend right, "q"] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Therefore, by Theorem 3, the square on the left is a pullback. This pullback should be $(f')^{-1}(p)$, hence $(f')^{-1}(p)$ and $f^{-1}(q)$ are homeomorphic to each other. Thus the fibers of f' are compact.

Next we show that f' is a closed map. At this point in the story of perfect maps, we somehow get through it by the power of general topology.

$$X' = \{\, (x, a) \in X \times Y' \mid f(x) = g(a) \,\} \tag{2.5}$$

and note that $f'\colon X \to Y'$ is the projection. Let A be a closed subset of X'. Take $p \notin f'(A)$. Note that $p \in Y'$. Then

$$(f^{-1}(g(p)) \times \{p\}) \cap A = \emptyset. \tag{2.6}$$

Indeed, if there were an intersection, it would contradict $p \notin f'(A)$. Since f is perfect, $f^{-1}(g(p))$ is compact, and hence $f^{-1}(g(p)) \times \{p\}$ is also compact. By the tube lemma (see Lemma 16 in Appendix), there exist open sets $V \subset X$ and $W \subset Y'$ such that $(V \times W) \cap A = \emptyset$ and $f^{-1}(g(p)) \times \{p\} \subset V \times W$. Now, since f is a closed map, if we put $O = Y \backslash f(X \backslash V)$, then this is an open subset of Y (this operation called the small image). Put $P = g^{-1}(O) \subset Y'$, and set $G = P \cap W$. We now prove that G is a neighborhood of p and does not meet $f'(A)$. First, let us show that P is a neighborhood of p. Since $f^{-1}(g(p)) \in V$, we have $f^{-1}(g(p)) \cap X \setminus V = \emptyset$. Hence $g(p) \notin f(X \setminus V)$. Therefore $g(p) \in$

$Y \setminus f(X \setminus V)$, so $p \in g^{-1}(Y \setminus f(X \setminus V)) = P$. Thus G is a neighborhood of p. Next we show that $G \cap f'(A) = \emptyset$. For $q \in G$, we have $q \in W$ and $g(q) \in Y \setminus f(X \setminus V)$. Thus $g(q) \notin f(X \setminus V)$, so $f^{-1}(g(q)) \cap (X \setminus V) = \emptyset$. That is, $f^{-1}(g(q)) \subset V$. Hence $f^{-1}(g(q)) \times \{q\} \subset V \times W$. Since $(V \times W) \cap A = \emptyset$, we have $q \notin f'(A)$. Thus G and $f(A)$ have empty intersection. By the arbitrariness of p, the complement of $f'(A)$ is open, so $f'(A)$ is closed. In other words, f' is a closed map.

Proof of (P3) $\Longrightarrow$ (P2): Consider the following diagram. By (A2) of Proposition 2, this is a pullback. Therefore, by (P3), the map $f \times 1_Z$ is perfect.

$$\begin{tikzcd} X \times Z \arrow[r, "pr"] \arrow[d, "f \times 1_Z"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \times Z \arrow[r, "pr"] & Y \end{tikzcd}$$

The implication (P2) $\Longrightarrow$ (P4) is obvious.

The implication (P3) $\Longrightarrow$ (P5) is obvious.

Proof of (P5) $\Longrightarrow$ (P4): This is the same as the proof of (P3) $\Longrightarrow$ (P2). Namely, consider the following diagram. By (A2) of Proposition 2, this is a pullback. Therefore, by (P5), the map $f \times 1_Z$ is closed.

$$\begin{tikzcd} X \times Z \arrow[r, "pr"] \arrow[d, "f \times 1_Z"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \times Z \arrow[r, "pr"] & Y \end{tikzcd}$$

Proof of (P4) $\Longrightarrow$ (P1): Taking $Z = \{*\}$, we see that f is a closed map. Next we prove that $f^{-1}(p)$ is compact. Let S be an arbitrary topological space.

$$\begin{tikzcd} f^{-1}(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \{*\} \arrow[r, "p"] & Y \end{tikzcd}$$

The above diagram is a pullback, so the following diagram is also a pullback (by (A5) of Proposition 2).

$$\begin{tikzcd} f^{-1}(p) \times S \arrow[r, "h"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times S \arrow[d, "f \times 1_S"] \\ S \arrow[r, "(p, 1_S)"] & Y \times S \end{tikzcd}$$

By Proposition 5 and by the fact that $(p, 1_S)$ is a closed embedding, the map h is also a closed embedding. Moreover, by (P4), the map $f \times 1_S$ is also closed. We now show that v is a closed map. If A is a closed subset of the upper-left space, then $\{p\} \times v(A) = (f \times 1_S)(h(A))$. By what we have accumulated so far, the right-hand side is closed.

Thus $\{p\} \times v(A)$ is closed in the subspace $\{p\} \times S$. Hence $v(A)$ is a closed subset of S, and it follows that v is a closed map. Therefore, by the arbitrariness of S and by (K3) of Proposition 8, the fiber $f^{-1}(p)$ is compact. Thus f is a perfect map. This completes the proof. $\square$

Corollary 10. *Let $f\colon X \to Y$ be a perfect map. Then for every subset B of Y, the restricted map $f|_{f^{-1}(B)}\colon f^{-1}(B) \to B$ is perfect.*

Proof. Consider the following diagram. Here $v = f|_{f^{-1}(B)}$.

$$\begin{tikzcd}
f^{-1}(B) \arrow[r, "h'"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
B \arrow[r, "h"] & Y
\end{tikzcd}$$

Since this diagram is a pullback, v is a perfect map. $\square$

Lemma 11. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be perfect maps. Then $g \circ f\colon X \to Z$ is also perfect.*

Proof. Take an arbitrary $h\colon A \to Z$ and consider the following diagram.

$$\begin{tikzcd}
P \arrow[r, "u'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Q \arrow[r, "u"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
A \arrow[r, "h"] & Z
\end{tikzcd}$$

Then, by (P5) of Theorem 9, f' and g' are closed maps. Hence their composite $g' \circ f'$ is also a closed map.

Therefore, in the pullback diagram

$$\begin{tikzcd}
P \arrow[r, "u'"] \arrow[d, "g' \circ f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g \circ f"] \\
Q \arrow[r, "u"] & Z
\end{tikzcd}$$

the map $g' \circ f'$ is always closed, and so, again by 9, the map $g \circ f$ is perfect. $\square$

Corollary 12. *Let $f\colon X \to Y$ be a perfect map, and let K be a compact subset of Y. Then $f^{-1}(K)$ is a compact subset of X.*

Proof. Put $g = f|_{f^{-1}(K)}\colon f^{-1}(K) \to K$. This is a perfect map by Corollary 10. Also $K \to \{*\}$ is a perfect map by (K2) of Proposition 8. By Lemma 11, the composition of these two maps, $f^{-1}(K) \to \{*\}$, is perfect. Therefore, again by (K2) of Proposition 8, $f^{-1}(K)$ is compact. $\square$

Theorem 13 ([7, Theorem 1.1]). *Let X, Y be topological spaces, and let Z be a Hausdorff space. Let $f\colon X \to Y$ be a perfect map and let $g\colon X \to Z$ be a continuous map. Then $(f, g)\colon X \to Y \times Z$ is a perfect map.*

Proof. The map $(1_X, g)\colon X \to X \times Z$ is a closed embedding because Z is Hausdorff (Proposition 6). In particular, it is perfect. Moreover, the map $f \times 1_Z\colon X \times Z \to Y \times Z$ is perfect ((P2) of Theorem 9). Therefore their composite, (f, g), is also a perfect map (Lemma 11). $\square$

Theorem 14 ([7, Corollary 1.4]). *Let A, B, C be topological spaces, and assume that B is Hausdorff. Let $\alpha\colon A \to B$ be a continuous map and let $\beta\colon B \to C$ be a perfect map. If $\beta \circ \alpha\colon A \to C$ is a perfect map, then α is also a perfect map.*

Proof. I could not think of a proof using pullbacks, so I give an ordinary proof. Put $\gamma = (\alpha, \beta \circ \alpha)\colon A \to B \times C$. Since B is Hausdorff, the map γ is perfect (Theorem 13). Let $G_\beta \subset B \times C$ be the graph of β. Then $pr\colon G_\beta \to B$ is a homeomorphism. In particular, it is perfect. Since $\gamma(A) \subset G_\beta$ and $\alpha = pr \circ \gamma$, it follows that α is perfect. $\square$

In this note, when we say a space is completely regular, we mean it in the sense that assumes T_1. Also, the symbol βX denotes the Stone–Čech compactification of a topological space X.

Theorem 15. *Let X and Y be completely regular spaces, and let $f\colon X \to Y$ be a continuous map. Then the following are equivalent.*

(Q1) *the map f is perfect.*
(Q2) *There exist a compact space K and a closed embedding $i\colon X \to Y \times K$ such that $f = pr \circ i$. Here $pr\colon Y \times K \to Y$.*
(Q3) *The following diagram is a pullback. Here g and g' are the natural embeddings.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ \beta X \arrow[r, "\beta f"] & \beta Y \end{tikzcd}$$

(Q4) *Let an arbitrary compactification of Y be denoted by ηY. Then the following diagram is a pullback.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ \beta X \arrow[r, "\beta f"] & \eta Y \end{tikzcd}$$

(Q5) *Let γX and ηY be compactifications of X and Y, respectively, which moreover make the following diagram commute. Then* *the following diagram is a pullback.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
\gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}$$

(Q6) *In the same situation as condition* (Q3)*, we have*

$$(\beta f)^{-1}(Y) = X.$$

(Q7) *In the same situation as condition* (Q4)*, we have*

$$(\beta f)^{-1}(Y) = X.$$

(Q8) *In the same situation as condition* (Q5)*, we have*

$$(\gamma f)^{-1}(Y) = X.$$

Proof. Proof of (Q1) $\implies$ (Q2): Consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, hook, "\Delta"] \arrow[dr, "f'\circ\Delta"'] & X \times \beta X \arrow[r, "pr"] \arrow[d, "f' = f \times 1_{\beta X}"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& Y \times \beta X \arrow[r, "pr"] & Y
\end{tikzcd}$$

The morphism extending from X to X becomes the identity map when composed. Thus, taking $K = \beta X$, (Q2) is satisfied.

Proof of (Q2) $\implies$ (Q1): This follows from (K3) of Proposition 8 and Lemma 11.

Proof of (Q3) $\vee$ (Q4) $\vee$ (Q5) $\implies$ (Q1): Maps such as βf and γf are continuous maps between compact Hausdorff spaces, hence are perfect maps. The conclusion follows from this and (P3) of Theorem 9.

Proof of (Q1) $\implies$ (Q3) $\wedge$ (Q4) $\wedge$ (Q5):

Since (Q5) $\implies$ (Q3) $\wedge$ (Q4), it is enough to prove (Q1) $\implies$ (Q5).

Consider the following diagram. After constructing Q, we construct P.

$$\begin{tikzcd}
P \arrow[r, "h'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Q \arrow[r, "h"] \arrow[d, hook, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "g"] \\
\gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}$$

Since $g\colon Y \to \eta Y$ is an embedding, the space Q is homeomorphic to $(\gamma f)^{-1}(Y)$. Thus we obtain the following diagram.

$$
\begin{tikzcd}
P \arrow[r, "h'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
(\gamma f)^{-1}(Y) \arrow[r, "h"] \arrow[d, hook, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "g"] \\
\gamma X \arrow[r, "\gamma f"] & \beta Y
\end{tikzcd}
$$

Here u' is the inclusion map.

By (P3) of the preceding Theorem 9, the map f' is perfect. Similarly, the map γf is perfect, so the map h is perfect, and h' is also perfect.

Since there is the obvious embedding $v'\colon X \to \beta X$, the following diagram holds.

$$
\begin{tikzcd}
X \arrow[drr, bend left, "1_X"] \arrow[ddddr, bend right, "v'"'] & & \\
& P \arrow[r, "h'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& (\gamma f)^{-1}(Y) \arrow[r, "h"] \arrow[d, hook, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "g"] \\
& \gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}
$$

Therefore, by the universal property of Q, there exists a unique $r\colon X \to Q$.

$$
\begin{tikzcd}
X \arrow[drr, bend left, "1_X"] \arrow[ddr, "r"] \arrow[dddr, bend right, "v'"'] & & \\
& P \arrow[r, "h'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& (\gamma f)^{-1}(Y) \arrow[r, "h"] \arrow[d, hook, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "v"] \\
& \gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}
$$

Furthermore, by the universal property of P, there exists a unique $l\colon X \to P$.

$$\begin{tikzcd}
X \arrow[dr, "l"] \arrow[drr, bend left, "1_X"] \arrow[ddr, "r"'] \arrow[dddr, bend right, "v'"'] & & \\
& P \arrow[r, "h'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& (\gamma f)^{-1}(Y) \arrow[r, "h"] \arrow[d, hook, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "v"] \\
& \gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}$$

Here $g' \circ l = 1_X$, and g' and 1_X are perfect maps. Therefore, by Theorem 14, the map l is also perfect. Furthermore, since $r = f' \circ l$, we see that r is also a perfect map.

Now $X \subset (\gamma f)^{-1}(Y)$, so there is an inclusion map $X \to (\beta f)^{-1}(Y)$. If we replace r by this map in the diagram, the commutative diagram is still satisfied. Thus, by the uniqueness of r, the map r is the inclusion map $X \to (\gamma f)^{-1}(Y)$. Since X is dense in γX, it is in particular dense in $(\gamma f)^{-1}(Y)$. We have now seen that r is a perfect map, hence in particular a closed map, so X is a closed subset of $(\gamma f)^{-1}(Y)$. By density, we see that $X = (\gamma f)^{-1}(Y)$. As a result, we conclude that r is the identity map. Moreover, by commutativity of the diagram, $h = f$. Thus we finally obtain the following diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook] \\
\gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}$$

This means that (Q5) holds.

Proof of (Q5) $\implies$ (Q6) $\wedge$ (Q7) $\wedge$ (Q8): Since (Q8) $\implies$ (Q7) $\wedge$ (Q8) holds, it is enough to prove (Q5) $\implies$ (Q8). This is similar to the proof above of (Q1) $\implies$ (Q5), but let us explain it anyway instead of being lazy.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook] \\
\gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}$$

is a pullback, and the diagram:

$$\begin{tikzcd}
(\gamma f)^{-1}(Y) \arrow[r, "(\gamma f|_{f^{-1}(Y)})"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook] \\
\gamma X \arrow[r, "\gamma f"] & \eta Y
\end{tikzcd}$$

is also a pullback. It follows that $X = (\gamma f)^{-1}(Y)$.

Proof of (Q6) $\vee$ (Q7) $\vee$ (Q8) $\Longrightarrow$ (Q1): This follows from the fact that βf and γf are perfect maps and from Corollary 10.

This completes the proof. $\square$

Remark. *The conditions* (Q6)*,* (Q7)*, and* (Q8) *of Theorem 15 are respectively equivalent to*

(1) $\beta f(\beta X \setminus X) = \mathrm{cl}_{\beta Y}(f(X)) \setminus f(X)$,
(2) $\beta f(\beta X \setminus X) = \mathrm{cl}_{\eta Y}(f(X)) \setminus f(X)$,
(3) $\gamma f(\gamma X \setminus X) = \mathrm{cl}_{\eta Y}(f(X)) \setminus f(X)$.

When f is surjective, these become

(1) $\beta f(\beta X \setminus X) = \beta Y \setminus Y$,
(2) $\beta f(\beta X \setminus X) = \eta Y \setminus Y$,
(3) $\gamma f(\gamma X \setminus X) = \eta Y \setminus Y$.

In other words, the image of the remainder of the compactification is equal to the remainder of the target.

3. Appendix

Lemma 16 (The tube lemma)**.** *Let X be a topological space, K be a compact space, $x \in X$ and U be an open set of $X \times K$. If U contains $\{x\} \times K$, then there exists a neighborhood N of x such that $N \times K \subset U$.*

Proof. Since K is compact and U is open, we can take finite open neighborhoods $O_1, \dots, O_n$ of x and open subsets $V_1, \dots, V_n$ such that

$$\{x\} \times K \subset \bigcup_{i=1}^{n} O_i \times V_i \subset U. \tag{3.1}$$

Puting $N = \bigcap_{i=1}^{n} O_i$ finishes the proof. $\square$

A continuous map f is called a **proper map** if the inverse image under f of every compact set is compact.

A space X is called a ***k*-space** if, for a subset A of X, A is closed if and only if, for every compact subset K, $K \cap A$ is closed in K. In other words, the quotient topology induced by the family of maps $\{\iota_K \colon K \to X\}_{K:\text{cpt in } X}$ agrees with the topology of X itself. Such a space is also called **compactly generated**.

A topological space X is called **weakly Hausdorff** if every compact subset is closed.

Proposition 17. *Let X be a topological space, and let Y be a compactly generated weakly Hausdorff space. Then a continuous map $f \colon X \to Y$ is a perfect map if and only if it is a proper map.*

Proof. First assume that f is perfect. Then, by Corollary 12, f is a proper map. Conversely, assume that f is proper. Then, by the definition of a proper map, each fiber of f is compact. We now show that f is a closed map. Let A be an arbitrary closed subset of X. It is enough to show that $f(A)$ is closed in Y. Let K be an arbitrary compact subset of Y. Then

$$f(A) \cap K = f(A \cap f^{-1}(K)) \tag{3.2}$$

holds. (This formula is called the projection formula, and it always holds for a map between sets and their subsets.) Since f is proper, $f^{-1}(K)$ is compact, and since A is closed in X, the set $A \cap f^{-1}(K)$ is closed in $f^{-1}(K)$. In particular, it is compact. Therefore $f(A \cap f^{-1}(K))$ is also compact. Thus $f(A) \cap K$ is a compact subset of Y. Since Y is weak Hausdorff, $f(A) \cap K$ is closed. By the arbitrariness of K and the fact that Y is a k-space, $f(A)$ is closed. Hence f is a closed map, and consequently a perfect map. □

Remark. *In* [1]*, Tomoki Yuji explained to me the properties of perfect maps with pullbacks, and every space was assumed to be completely regular. Perhaps the category of completely regular spaces has a somewhat scheme-like flavor. In the present document, I have modified the argument so that, except for the final theorem, the proofs are carried out as much as possible for general topological spaces. As a result, the proof of Theorem 9 may have become a strange one in which the implications are proved one by one.*

Remark. *According to* [7]*,* (Q2) *of Theorem 15 seems to have been found in* [6, Theorem 1] *and* [8, Theorem, p.21]. *According to* [5]*,* (Q3) *of Theorem 15 seems to have been found in* [4, Lemma 1.5].

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