https://yoshito-ishiki-math.github.io

A CHEATSHEET ON NON-ARCHIMEDEAN ANALOGUES IN METRIC GEOMETRY

YOSHITO ISHIKI

俺はゴゴ。ずっと物まねをして生きてきた。お前達は久しぶりの来客だ。
そうだ。お前達の物まねをしてやろう。お前達は今何をしているんだ？
そうか。世界を救おうとしているのか。
では俺も世界を救うという物まねをしてみるとしよう。

ものまね士ゴゴ: ファイナルファンタジー6

I am Gogo, master of mimicry. It has been a long, long time since anyone visited me here...
I have been idle for too many years... Perhaps I ought to mimic you. Tell me, what are you doing here?
I see... So you seek to save the world.
Then I guess that means that I shall save the world as well. Lead on! I will copy your every move.

Gogo: Final Fantasy VI, English version of the above epigraph.

Date: March 25, 2024.
Key words and phrases. Ultrametrics, Non-Archimedean analogues.

Abstract

In contrast to conventional mathematics, there exists a domain that could be referred to as "zero-dimensional mathematics". A vivid example of this is "non-Archimedean functional analysis" based on the field $\mathbb{Q}_p$ of p-adic numbers, as opposed to conventional functional analysis based on the field $\mathbb{R}$ of real numbers. Although this zero-dimensional world acts as a reflective mirror to our familiar universe, we can find both of Archimedean phenomena that cannot be translated into zero dimensions, and non-Archimedean phenomena that are beyond the expectations of conventional mathematics. In this note, we explain some non-Archimedean analogues in metric geometry.

1. Table of analogues

In this section, we give a table of analogues between Archimedean mathematics and non-Archimedean one. The left hand side is corresponding to Archimedean things, and the other side is for non-Archimedean things.

Archimedean things	**Non-Archimedean things**
Analysis	Non-Archimedean analysis (p-adic analysis). See [5] and [29].
Functional analysis	Non-Archimedean functional analysis. See [1], [32], [24], and [29].
Hahn fields (see [26], [19] and [6]).	p-adic Hahn fields. The author calls them the Poonen fields (see [26] and [6]).
Regular spaces	Ultraregular spaces, which means that the spaces have **small** inductive dimension 0.
Normal spaces.	Ultranormal spaces, which means that the spaces have **large** inductive dimension 0.

Paracompact spaces.	Ultraparacompact spaces (see [9]). A space X is ultraparacompact if and only if X is paracompact and has covering dimension 0.
The Hilbert cube $[0,1]^{\aleph_0}$.	The Cantor set $\{0,1\}^{\aleph_0}$.
The space $\mathbb{R}^{\aleph_0}$ of sequences of reals	The space $\omega_0^{\aleph_0}$ of sequences of integers which also coincides with the space of irrationals.
Paracompactness and normality: **Theorem 1.1.** *Every paracompact Hausdorff space is normal.*	Ultraparacompactness and ultranormality: **N.-A. Theorem 1.1.** *Every ultraparacompact Hausdorff space is ultranormal.*
Michael's continuous selection theorem([22])	0-dimensional Michael's continuous selection theorem ([22] and [23]).
The operator $+$: the additional operator on $\mathbb{R}$	The operator $\vee$: the maximal operator on $\mathbb{R}$. Namely, $x \vee y = \max\{x, y\}$.
The triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$.	The strong triangle inequality: $d(x,y) \leq d(x,z) \vee d(z,y)$.
The space $[0,\infty)$: the set of non-negative real numbers. In this note we consider that all metrics take values in $[0,\infty)$. Of course, we can find research on metric taking values in some restricted subset of $[0,\infty)$.	The space R: a range set, a set s.t. $0 \in \mathbb{R}$ and $R \subseteq [0,\infty)$. It is often convenient to consider R-valued non-Archimedean analogues rather than only non-Archimedean analogues.

A metric : $d\colon X \times X \to [0, \infty)$.	An R-valued ultrametric $d\colon X \times X \to R$. For some reasons, when considering ultrametrics, it is useful to treat R-valued ultrametrics for a range set R, instead of ($[0, \infty)$-valued) ultrametrics.
The Stone theorem on paracompactness: **Theorem 1.2** ([30] and [27]). *Every metrizable space is paracompact.*	The non-Archimedean Stone theorem on paracompactness: **N.-A. Theorem 1.2.** *Every ultrametrizable space is ultraparacompact.* For the proof, we refer the readers to [9, Proposition 1.2 and Corollary 1.4] and [7, Theorem II].
The Euclidean metric $\lvert x - y \rvert$. This can be represented as $\lvert x - y \rvert = \inf\{\epsilon \in \mathbb{R} \mid x \leq y + \epsilon \text{ and } y \leq x + \epsilon\}$.	The non-Archimedean Euclidean metric M_R on R defined by $M_R(x, y) = \inf\{\epsilon \in \mathbb{R} \mid x \leq y \vee \epsilon \text{ and } y \leq x \vee \epsilon\}$. This metric also can be represented as $M_R(x, y) = x \vee y$ if $x \neq y$; otherwise, $M_R(x, y) = 0$.
Theory of retracts (of metric spaces).	There is no theory of retracts of ultrametric spaces because every non-empty closed subsets of an ultrametric space is a (Lipschitz) retract of the ambient space ([4]). That is to say, in a category of ultrametric spaces, all extension problems are trivial.

The Banach–Mazur theorem:	A non-Archimedean Banach-Mazur theorem:
Theorem 1.3. *The space $C([0,1],\mathbb{R})$ is universal for all separable metric spaces. Namely, every separable metric space can be isometrically embedded into $C([0,1],\mathbb{R})$.* Note that this theorem is often proven by showing that $C(\Gamma,\mathbb{R})$ is universal for all separable metric spaces.	**N.-A. Theorem 1.3** ([16])**.** *The space $\mathrm{C}_0(\Gamma, R)$ is universal for all separable R-valued ultrametric spaces. Namely, every separable R-valued ultrametric space can be isometrically embedded into $\mathrm{C}_0(\Gamma, R)$.* Note that the space $\mathrm{C}_0(\Gamma, R)$ does not have an algebraic structure.
The space of metrics: $\mathrm{Met}(X)$. The set $\mathrm{Met}(X)$ is defined as the set of all metrics on X that generate the same topology of X.	The space of R-valued ultrametrics: $\mathrm{UMet}(X;R)$. The set $\mathrm{UMet}(X;R)$ is defined as the set of all R-valued ultrametrics on X that generate the same topology of X.
The supremum metric $\mathcal{D}_X$ on $\mathrm{Met}(X)$ defined by $\mathcal{UD}_X(d,e) = \sup_{x,y\in X} \lvert d(x,y) - e(x,y)\rvert$. This coincides with the infimum of all $\epsilon \in (0,\infty)$ such that $d(x,y) \le e(x,y) + \epsilon$ and $e(x,y) \le d(x,y) + \epsilon$.	The metric $\mathcal{UD}_X^R$ on $\mathrm{UMet}(X;R)$ defined by declaring that $\mathcal{UD}_X(d,e)$ is the infimum all $\epsilon \in (0,\infty)$ such that $d(x,y) \le e(x,y) \vee \epsilon$ and $e(x,y) \le d(x,y) \vee \epsilon$.

The Arens–Eells theorem: **Theorem 1.4** ([3])**.** *For every metric space* (X, d) *there exist a a normed space* $(V, \|\ast\|)$ *over* $\mathbb{R}$ *and an isometric embedding* $I\colon X \to V$ *such that the image* $I(X)$ *is closed and linearly independent in* V*.*	A candidate of a non-Archimedean Arens–Eells theorem: **N.-A. Theorem 1.4** ([14, Thm 1.1])**.** *Let* $(A, \vert \ast \vert)$ *be an integral ring equipped with the trivial absolute value, i.e.,* $\vert x\vert = 1$ *if* $x \neq 0$*. Let* R *be a range set. Then for every* R*-valued ultrametric space* (X, d) *there exist a an ultra-normed space* $(V, \|\ast\|)$ *over* A *and an isometric embedding* $I\colon X \to V$ *such that the image* $I(X)$ *is closed and linearly independent in* V*.*
The Kuratowski embedding $X \to C(X)$.	A candidate of a non-Archimedean Kuratowski embedding is the Schikhof embedding $X \to K$ (see [28]), where K is a large non-Archimedean valued field.

The Hausdorff metric extension theorem: **Theorem 1.5** ([12])**.** *Let X be a metrizable space, and A be a closed subset of X. The for every $d \in \mathrm{Met}(A)$, there exists $D \in \mathrm{Met}(X)$ such that $D\|_{A\times A} = d$. Moreover, if X is completely metrizable, and d is complete, then we can choose D as a complete one.*	A non-Archimedean Hausdorff extension(An ultrametric extension theorem): **N.-A. Theorem 1.5** ([14])**.** *Let R be a range set, X be a metrizable space, and A be a closed subset of X. The for every $d \in \mathrm{UMet}(A; R)$, there exists $D \in \mathrm{UMet}(X; R)$ such that $D\|_{A\times A} = d$. Moreover, if X is completely metrizable, and d is complete, then we can choose D as a complete one.*
Hyperspace. For a metric space (X, d), the space of all compact subsets of X with the Hausdorff distance is called the hyperspace of (X, d).	Hyperspace. Since the Hausdorff metric of an ultrametric becomes an ultrametric, the construction of hyperspaces is a non-Archimedean analogue of itself.
The Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$. This space is a moduli space of all compact metric space equipped with the Gromov–Hausdorff distance $\mathcal{GH}$.	The non-Archimedean Gromov–Hausdorff space $(\mathcal{U}_R, \mathcal{NA})$. This space is a moduli space of all R-valued compact ultrametric space equipped with the R-valued non-Archimedean Gromov–Hausdorff distance $\mathcal{NA}$.
Strongly rigid metrics (see [18]).	There is no non-Archimedean analogues of strongly rigid metrics because all triangles in ultrametrics are isosceles.
$\mathcal{F}$: the class of all finite metric spaces	$\mathcal{N}(R)$: the class of all finite R-valued ultrametric spaces
$\mathcal{F}$-injectivity	$\mathcal{N}(R)$-injectivity

The Urysohn universal space $(\mathbb{U}, \rho)$. This space is a separable complete $\mathcal{F}$-injective metric space. Such a space is unique up to isometry. See [13], [20], [21], [25], and [31].	The Urysohn universal ultrametric space $(\mathbb{V}_R, \sigma_R)$. When R is countable, this space is a separable R-valued $\mathcal{N}(R)$-injective ultrametric space. When R is uncountable, it is defined as the R-petaloid ultrametric space. In any case, such a space is unique up to isometry. For the case where R is countable, see [10], [33], [16]. For the uncountable case, see [17]. See also [8].
The quotient metric space of the hyperspace of $(\mathbb{U}, \rho)$ by $\mathrm{Isom}(\mathbb{U}, \rho)$ is isometric to the Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$. (see [11, Exercise (b) in the page 83], [2, Theorem 3.4], and [33]).	The R-valued Gormov–Hausdorff ultrametric space is isometric to $(\mathbb{V}_R, \sigma_R)$. See [33] and [17].
Products: For any $p \in [0, \infty]$, it is true that $(\mathbb{U} \times \mathbb{U}, \rho \times_p \rho) \underset{\mathrm{Isom}}{\not\equiv} (\mathbb{U}, \rho)$ (see [15]).	Products: It is true that $(\mathbb{V}_R \times \mathbb{V}_R, \sigma_R \times_\infty \sigma_R) \underset{\mathrm{Isom}}{\equiv} (\mathbb{V}_R, \sigma_R)$ (see [15])
Hyperspaces: It is **unknown** whether $(\mathcal{K}(\mathbb{U}), \mathcal{HD}_\rho) \underset{\mathrm{Isom}}{\equiv} (\mathbb{U}, \rho)$ or not. The author thinks this is negative.	Hyperspaces: It is true that $(\mathcal{K}(\mathbb{V}_R), \mathcal{HD}_{\sigma_R}) \underset{\mathrm{Isom}}{\equiv} (\mathbb{V}_R, \sigma_R)$ (see [15]).

References

[1] N. L. Alling, *Foundations of analysis over surreal number fields*, North-Holland Mathematics Studies, vol. 141, North-Holland Publishing Co., Amsterdam, 1987, Notas de Matemática, 117. [Mathematical Notes]. MR 886475

[2] S. A. Antonyan, *The Gromov–Hausdorff hyperspace of a Euclidean space*, Adv. Math. **363** (2020), 106977, 30, DOI:10.1016/j.aim.2020.106977. MR 4052556

[3] R. F. Arens and J. Eells, Jr., *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397–403. MR 81458

[4] N. Brodskiy, J. Dydak, J. Higes, and A. Mitra, *Dimension zero at all scales*, Topology Appl. **154** (2007), no. 14, 2729–2740. MR 2340955

[5] A. B. Comicheo, *Generalized open mapping theorem for X-normed spaces*, p-Adic Numbers Ultrametric Anal. Appl. **11** (2019), no. 2, 135–150, DOI:10.1134/S2070046619020043. MR 3946544

[6] A. B. Comicheo and K. Shamseddine, *Summary on non-Archimedean valued fields*, Advances in ultrametric analysis (A. Escassut, C. Perez-Garcia, and K. Shamseddine, eds.), Contemp. Math., vol. 704, American Mathematical Society, Providence, RI, 2018, pp. 1–36, DOI:10.1090/conm/704/14158. MR 3782290

[7] J. de Groot, *Non-Archimedean metrics in topology*, Proc. Amer. Math. Soc. **7** (1956), 948–953, DOI:10.2307/2033568. MR 80905

[8] C. Delhomme, C. Laflamme, M. Pouzet, and N. Sauer, *On homogeneous ultrametric spaces*, arXiv preprint arXiv:1509.04346 (2015).

[9] R. L. Ellis, *Extending continuous functions on zero-dimensional spaces*, Math. Ann. **186** (1970), 114–122, DOI:10.1007/BF01350686. MR 261565

[10] S. Gao and C. Shao, *Polish ultrametric Urysohn spaces and their isometry groups*, Topology Appl. **158** (2011), no. 3, 492–508, DOI:10.1016/j.topol.2010.12.003. MR 2754373

[11] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston, Inc., Boston, MA, 1999, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[12] F. Hausdorff, *Erweiterung einer homöomorphie*, Fund. Math. **16** (1930), 353–360.

[13] M. Hušek, *Urysohn universal space, its development and Hausdorff's approach*, Topology Appl. **155** (2008), no. 14, 1493–1501, DOI:10.1016/j.topol.2008.03.020. MR 2435145

[14] Y. Ishiki, *An embedding, an extension, and an interpolation of ultrametrics*, p-Adic Numbers Ultrametric Anal. Appl. **13** (2021), no. 2, 117–147, DOI:10.1134/S2070046621020023. MR 4265905

[15] ———, *Characterizations of Urysohn universal ultrametric spaces*, (2023), arXiv:2303.17471.

[16] ———, *Constructions of Urysohn universal ultrametric spaces*, (2023), arXiv:2302.00305, to appear in p-Adic Numbers Ultrametric Anal. Appl.

[17] ———, *Uniqueness and homogeneity of non-separable Urysohn universal ultrametric spaces*, (2023), arXiv:2302.00306.

[18] ———, *Strongly rigid metrics in spaces of metrics*, Topology Proc. **63** (2024), 125–148, arXiv:2210.02170.

[19] S. Kuhlmann, *Ordered exponential fields*, Fields Institute Monographs, vol. 12, American Mathematical Society, Providence, RI, 2000, DOI:10.1090/fim/012. MR 1760173

[20] J. Melleray, *On the geometry of Urysohn's universal metric space*, Topology Appl. **154** (2007), no. 2, 384–403, DOI:10.1016/j.topol.2006.05.005. MR 2278687

[21] ______, *Some geometric and dynamical properties of the Urysohn space*, Topology Appl. **155** (2008), no. 14, 1531–1560, DOI:10.1016/j.topol.2007.04.029. MR 2435148

[22] E. Michael, *Continuous selections. I*, Ann. of Math. (2) **63** (1956), 361–382, DOI:10.2307/1969615. MR 77107

[23] ______, *Selected Selection Theorems*, Amer. Math. Monthly **63** (1956), no. 4, 233–238, DOI:10.2307/2310346. MR 1529282

[24] H. Ochsenius and W. H. Schikhof, *Banach spaces over fields with an infinite rank valuation*, p-adic functional analysis (Poznań, 1998), Lecture Notes in Pure and Appl. Math., vol. 207, Dekker, New York, 1999, pp. 233–293. MR 1703500

[25] V. Pestov, *Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, Israel J. Math. **127** (2002), 317–357, DOI:10.1007/BF02784537. MR 1900705

[26] B. Poonen, *Maximally complete fields*, Enseign. Math. (2) **39** (1993), no. 1-2, 87–106. MR 1225257

[27] M. E. Rudin, *A new proof that metric spaces are paracompact*, Proc. Amer. Math. Soc. **20** (1969), 603, DOI:10.2307/2035708. MR 236876

[28] W. H. Schikhof, *Isometrical embeddings of ultrametric spaces into non-Archimedean valued fields*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), no. 1, 51–53. MR 748978

[29] ______, *Ultrametric Calculus*, Cambridge Studies in Advanced Mathematics, vol. 4, Cambridge University Press, Cambridge, 2006, Reprint of the 1984 original [MR0791759]. MR 2444734

[30] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. **54** (1948), 977–982, DOI:10.1090/S0002-9904-1948-09118-2. MR 26802

[31] P. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. **51** (1927), 43–64 and 74–90.

[32] A. C. M. van Rooij, *Non-Archimedean functional analysis*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 51, Marcel Dekker, Inc., New York, 1978. MR 512894

[33] Z. Wan, *A novel construction of Urysohn universal ultrametric space via the Gromov–Hausdorff ultrametric*, Topology Appl. **300** (2021), Paper No. 107759, DOI:10.1016/j.topol.2021.107759. MR 4282005

(Yoshito Ishiki) HTTPS://YOSHITO-ISHIKI-MATH.GITHUB.IO