THE TOPOLOGY OF GROMOV–HAUSDORFF SPACE

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Abstract. This paper is divided into four parts. We prove that the Gromov–Hausdorff space is homeomorphic to the Hilbert space. In Part I, we construct an assignment of a full-support probability measure to every nonempty compact metric space that respects isometries and is continuous for simultaneous Hausdorff convergence of the spaces and weak convergence of the measures. In Part II, we use these measures to construct finite-dimensional local models whose induced pseudometrics approximate the original distances uniformly and whose norms and point maps vary continuously up to orthogonal changes of coordinates. In Part III, we use the local models to prove that the Gromov–Hausdorff space is an absolute retract for all metrizable spaces. In Part IV, we establish a discrete approximation property and conclude that the space of isometry classes of nonempty compact metric spaces is homeomorphic to the real separable infinite-dimensional Hilbert space.

Contents

Key words and phrases. Gromov–Hausdorff space, Hilbert space, absolute retract, probability measure, hyperspace.

1. Introduction

The Gromov–Hausdorff distance permits compact metric spaces with different underlying sets to be compared in a single metric space. Let $\mathcal{M}$ denote the space of isometry classes of nonempty compact metric spaces with distance $\mathcal{GH}$. Its points range from finite spaces to compact spaces of arbitrary dimension. We study the topological type of $\mathcal{M}$. Antonyan records the questions whether $\mathcal{M}$ is an absolute retract and whether it is homeomorphic to ℓ^2 in [5, p. 2]. For each positive integer n, Antonyan identifies the subspace represented by compact subsets of $\mathbb{R}^n$ with the Hilbert cube minus a point [6, Corollary 5.3].

The author constructed families of branching Gromov–Hausdorff geodesics continuously parametrized by the Hilbert cube [20, Theorem 1.3]. The author also constructed topological embeddings of arbitrary compact metrizable spaces, with finitely many distinct prescribed values, into the subsets of $\mathcal{M}$ consisting of continua [21, Theorem 1.1], spaces with prescribed dimensions [22, Theorem 1.3], and compact metric trees [23, Theorem 1.1], respectively.

The subspaces of $\mathcal{M}$ consisting of finite metric spaces also have a precise dimension theory. Nakajima, Yamauchi, and Zava [32, arXiv v1, Theorem A] prove that the subspace of spaces with at most n points has topological dimension $n(n-1)/2$. Nakajima and Shioya [31, arXiv v1, Main Theorem 1.2] construct a compact metrizable space containing a dense topological copy of $\mathcal{M}$. For metric measure spaces, Kazukawa, Nakajima, and Shioya [26, arXiv v1, Theorems 1.4–1.6] prove contractibility and local path connectedness for both the box and concentration topologies. Our classification concerns the ordinary unmeasured Gromov–Hausdorff topology.

The relation between Gromov–Hausdorff geometry and hyperspaces is useful for this question. Mémoli and Wan [29, arXiv v2, Theorem 1] prove that every Gromov–Hausdorff geodesic can be realized as a Hausdorff geodesic of compact subsets of one compact metric space. Curtis [12, Theorem 1.6] proves, in particular, that the hyperspace of nonempty compact subsets of a Banach space, equipped with the Hausdorff metric of its norm, is an absolute retract. In the present paper, we construct approximations to the identity of $\mathcal{M}$ through hyperspaces of nonempty compact subsets of Banach spaces, modulo compact groups. These approximations lead to the following classification.

Theorem 1.1 (Theorem 12.1)**.** *The space $\mathcal{M}$ is homeomorphic to the real Hilbert space ℓ^2.*

The proof establishes two topological properties. We first prove that $\mathcal{M}$ is an absolute retract for metrizable spaces. We then approximate maps from countably many compact domains so that their images form a discrete family. The combination satisfies the characterization of Hilbert space topology in [34, p. 248, assertion (i)], with the correction in [35, Section C].

The interpolation result in [25, Theorem 1.1] controls the uniform error when extending prescribed metrics from a discrete family of closed subsets of a fixed metrizable space. The absolute retract argument here requires compatible approximations of the metrics on varying compact spaces. For this purpose, we construct a full-support probability measure on each compact metric space, invariant under all its isometries. The measures depend continuously on the space for Hausdorff convergence of the underlying sets and weak convergence of the measures in a common compact ambient space. Finite spectral subspaces of the corresponding distance operators then approximate the distance functions. These operators and their spectral coordinates were studied by Maria, Oudot, and Solomon [28, Section 3]. For our local models, we approximate distance functions uniquely in finite L^p norms on whole spectral subspaces. The coordinate ambiguity is an orthogonal group, which can be retained in an orbit space. A partition of unity combines these local constructions into global approximations.

Multiplication by a sufficiently short interval and a sufficiently small finite equilateral space changes the Gromov–Hausdorff distance by a controlled amount. Raising the product distances to distinct powers distinguishes the resulting spaces by the Hausdorff dimensions of their connected subsets. Increasing the cardinalities of the finite factors prevents the images from accumulating in $\mathcal{M}$.

The main theorem also clarifies the local contraction question. We use local contractibility in the sense that every neighborhood U of a point contains a neighborhood V whose inclusion into U is null-homotopic. The homeomorphism in Theorem 1.1 implies the stronger existence of a basis of open neighborhoods which strongly deform onto their centers. This conclusion is topological and does not prescribe the neighborhoods as balls for $\mathcal{GH}$.

Organization. The introduction and Section 2 provide the common background for four parts. Each part also has its own preliminaries, in Sections 4, 7, 10, and 13, respectively. Part I, introduced in Section 3 and developed in Section 5, constructs continuous invariant probabilities. Part II, introduced in Section 6 and developed in Section 8, uses these probabilities to construct finite-dimensional local models. Part III, introduced in Section 9 and developed in Section 11, combines the models into global approximations and proves the absolute retract property. Part IV, introduced in Section 12 and developed in Section 14, proves the discrete approximation property and completes the proof of Theorem 1.1. Section 15 concludes with questions about related spaces and the geometry of GH balls.

Conventions and notation. All metric spaces (X, d) represented by points of $\mathcal{M}$ are nonempty and compact. We identify such a pair with its isometry class when it occurs in $\mathcal{M}$. The symbol X denotes its underlying set, with the topology induced by d. Subscripts on constructions such as $\mathfrak{m}_{X,d}$ refer to the specified pair (X, d). When isometric embeddings into (Z, h)

are specified, we use the same letters for the embedded subsets and their restricted metrics. The notation id_X denotes the identity map on X. We omit the subscript when the domain is specified. Banach and Hilbert spaces are real unless complex scalars are explicitly indicated. The notation $\mathcal{K}(X)$ denotes the set of nonempty compact subsets of X. For a specified metric d, we equip it with the Hausdorff metric $\mathsf{H}[d]$. The notation $\mathcal{P}\mathrm{rob}(S)$ denotes the Radon probability measures on a metrizable space S with the topology of weak convergence against bounded continuous functions. An isometric embedding $f\colon (X,d) \to (Y,e)$ is a map satisfying $e(f(x),f(y)) = d(x,y)$ for every $x,y \in X$. In this paper an isometry means a surjective isometric embedding. We use $\mathbb{Z}_{\geq 0} = \{0,1,2,\dots\}$ as the index set of sequences unless stated otherwise.

Use of AI. OpenAI Codex, based on GPT-6, was used to explore and develop the mathematical constructions and proofs, search the literature, and prepare the exposition and LaTeX source. The author has verified all mathematical arguments, understands the proofs and constructions, and takes full responsibility for the mathematical content and the final manuscript.

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2. Common preliminaries

We recall metric convergence and compactness, comparison of metric quotients, and weak convergence of probability measures. The final subsection fixes the contraction and absolute extension terminology.

2.1. Gromov–Hausdorff convergence and compactness.

We begin with the correspondence description of the distance and compactness criteria.

For a metric space (X,d), a point $x \in X$, and a radius $r > 0$, we write

$$B(x,r;d) = \{y \in X \mid d(x,y) < r\}.$$

For a subset $A \subset X$, we put

$$\mathrm{dist}_d(x,A) = \inf_{y \in A} d(x,y), \qquad \mathrm{dist}_d(x,\emptyset) = \infty.$$

We denote by $\mathcal{K}(X)$ the set of all nonempty compact subsets of X. The *Hausdorff distance* on this set is defined by

$$\mathsf{H}[d]\colon \mathcal{K}(X) \times \mathcal{K}(X) \to [0,\infty),$$
$$\mathsf{H}[d](S,T) = \max\left\{\sup_{x \in S} \mathrm{dist}_d(x,T), \sup_{y \in T} \mathrm{dist}_d(y,S)\right\}.$$

The *Vietoris topology* on $\mathcal{K}(X)$ has a basis consisting of the sets

$$\left\{S \in \mathcal{K}(X) \;\middle|\; S \subset \bigcup_{i=1}^{m} U_i, \quad S \cap U_i \neq \emptyset \ (1 \leq i \leq m)\right\},$$

where $m \geq 1$ and $U_1, \ldots, U_m$ are open subsets of X.

Theorem 2.1 ([8, Exercises 2.2.11(a), (b) and 3.2.9])**.** *Let (X, d) be a metric space. Then the Hausdorff distance induces the Vietoris topology on $\mathcal{K}(X)$. If X is separable, then $\mathcal{K}(X)$ is separable.*

A subset $A \subset X$ is an *ε-net*, where $\varepsilon > 0$, if

$$\forall x \in X \quad \exists z \in A \quad d(x, z) < \varepsilon.$$

It is a *finite ε-net* if it is finite. Every compact metric space admits a finite net at each positive scale.

A *correspondence* between nonempty sets X and Y is a subset of $X \times Y$ whose two coordinate projections are surjective. For metric spaces (X, d) and (Y, e), the distortion of a correspondence R is

$$\operatorname{dis}(R) = \sup \{ |d(x_0, x_1) - e(y_0, y_1)| \mid (x_0, y_0), (x_1, y_1) \in R \}.$$

Theorem 2.2 ([11, Theorem 7.3.25])**.** *For nonempty compact metric spaces (X, d) and (Y, e), the Gromov–Hausdorff distance satisfies*

$$\mathcal{GH}((X, d), (Y, e)) = \frac{1}{2} \inf_{R} \operatorname{dis}(R). \tag{2.1}$$

Equivalently, it is the infimum of the Hausdorff distances between isometric copies of (X, d) and (Y, e) in a common metric space.

The next construction controls each pair in a prescribed correspondence. We use it below to embed a convergent sequence of compact metric spaces into one compact metric space.

Lemma 2.3. *Let (X, d) and (Y, e) be nonempty compact metric spaces, let $R \subset X \times Y$ be a correspondence, and let $\eta > 0$ satisfy $\operatorname{dis}(R) \leq 2\eta$. Then there is a metric h on the disjoint union $X \sqcup Y$ extending both metrics such that*

$$h(x, y) \leq \eta \quad ((x, y) \in R).$$

In particular, $\mathsf{H}[h](X, Y) \leq \eta$.

Proof. We use the construction in [11, proof of Theorem 7.3.25, pp. 257–258]. Retain the original metrics on each component and set

$$h(x, y) = \inf_{(x_0, y_0) \in R} \big(d(x, x_0) + \eta + e(y_0, y) \big) \quad (x \in X, \ y \in Y).$$

Define the reverse distances by symmetry. Cross distances are at least η and are finite. The triangle inequalities with a cross distance on the left follow from those for the original metrics and the infimum formula. For the other type of triangle, take $x, x_1 \in X$, $y \in Y$, and $(x_0, y_0), (x_2, y_2) \in R$. The distortion bound implies

$$\begin{aligned} d(x, x_1) &\leq d(x, x_0) + d(x_0, x_2) + d(x_2, x_1) \\ &\leq d(x, x_0) + \eta + e(y_0, y) + e(y, y_2) + \eta + d(x_2, x_1). \end{aligned}$$

Taking the two infima proves the required inequality. Interchanging the components proves its counterpart. Finally, using the pair (x, y) itself in the infimum proves the asserted bound on matched points. $\square$

Theorem 2.4 (Consequence of [11, Proposition 7.4.12])**.** *If* $(X_j, d_j) \to (X, d)$ *in* $\mathcal{M}$*, then for every* $\varepsilon > 0$ *there is a positive integer* m *such that each* X_j *admits an* ε*-net of cardinality at most* m*.*

A subset of a metric space is called *c-separated* if the distance between any two distinct points is at least c.

Corollary 2.5. *Let* $(X_j, d_j) \to (X, d)$ *in* $\mathcal{M}$*, and let* $c > 0$*. Then there is a positive integer* m *such that every* c*-separated subset of every* X_j *has cardinality at most* m*.*

Proof. By Theorem 2.4, choose $c/3$-nets in all X_j with a common cardinality bound m. Each ball of radius $c/3$ contains at most one point of a c-separated subset. Thus the same bound applies to these subsets. $\square$

We use the following standard property of the Gromov–Hausdorff space $\mathcal{M}$.

Lemma 2.6 ([37, author's manuscript, p. 768])**.** *The metric space* $(\mathcal{M}, \mathcal{GH})$ *is complete and separable.*

Lemma 2.7. *Every Cauchy sequence* $\{(X_j, d_j)\}_{j \in \mathbb{Z}_{\geq 0}}$ *in* $\mathcal{M}$ *admits isometric embeddings into a compact metric space* (Z, h) *whose images converge in Hausdorff distance to a nonempty compact subset. If its Gromov–Hausdorff limit is* (X, d)*, then we may also embed* (X, d) *and obtain*

$$\mathsf{H}[h](X_j, X) \longrightarrow 0,$$

where the spaces are identified with their images.

Proof. By Lemma 2.6, the Cauchy sequence has a limit (X, d). Set

$$\varepsilon_j = \mathcal{GH}((X_j, d_j), (X, d)) + 2^{-j}.$$

By the correspondence formula in Theorem 2.2, choose a correspondence between each term and the limit with distortion less than $2\varepsilon_j$. By Lemma 2.3, we obtain a metric h_j on $X_j \sqcup X$ extending both metrics and satisfying

$$\mathsf{H}[h_j](X_j, X) \leq \varepsilon_j.$$

The metric gluing lemma [37, author's manuscript, Theorem 27.2] allows these couplings to be joined along their common copy of X. Each finite stage preserves the metrics already constructed, so the union carries a metric preserving all the embeddings and bounds. For every $\eta > 0$, there is an index j_0 such that X_j lies within distance $\eta/2$ of X for every $j \geq j_0$. A finite $\eta/2$-net in X therefore covers the tail by balls of radius η. The finitely many earlier compact spaces also admit finite nets. Thus the union is totally bounded, and its completion is the required compact ambient space. $\square$

We also record the compactness criterion used for families of functions.

Theorem 2.8 (Arzelà–Ascoli, [30, Theorem 6.26])**.** *Let (X,d) be nonempty and compact, and let (Y,e) be complete. Equip the set $C(X,Y)$ of continuous maps with the uniform metric*

$$d_\infty(f,g) = \sup_{x\in X} e(f(x),g(x)).$$

Then a family $\mathcal{F} \subset C(X,Y)$ has compact closure if and only if its combined image $\bigcup_{f\in\mathcal{F}} f(X)$ is totally bounded and the family $\mathcal{F}$ is equicontinuous. Here equicontinuity means that for every $\varepsilon > 0$ there is $\eta > 0$ such that

$$e(f(x),f(y)) < \varepsilon \quad \text{whenever } f \in \mathcal{F}, \quad x,y \in X, \quad d(x,y) < \eta.$$

The cited theorem characterizes total boundedness in the uniform metric. Completeness of the target (Y,e) implies completeness of the function space $C(X,Y)$ with the uniform metric d_∞, which yields the formulation in terms of compact closure above.

2.2. **Pseudometrics and metric quotients.** We record estimates that remain applicable when distinct points are identified.

The diameter is continuous on $\mathcal{M}$, since

$$|\mathrm{diam}(X,d) - \mathrm{diam}(Y,e)| \le 2\mathcal{GH}((X,d),(Y,e)).$$

A continuous pseudometric q on a compact space X defines the equivalence relation $x \sim_q y$ if $q(x,y) = 0$. Its metric quotient is

$$[X,q] = (X/{\sim_q}, \overline{q}), \qquad \overline{q}([x],[y]) = q(x,y).$$

This quotient is compact. When q is a metric, it is isometric to (X,q). The comparison of a pseudometric quotient with a metric on the same set appears in the author's [22, Proposition 2.12]. We use the following correspondence form, which also compares two metric quotients.

Lemma 2.9. *Let X,Y be nonempty compact spaces and $R \subset X \times Y$ a correspondence. For continuous pseudometrics q on X and r on Y, put*

$$\Delta_R(q,r) = \sup_{(x_0,y_0),(x_1,y_1)\in R} |q(x_0,x_1) - r(y_0,y_1)|.$$

Then

$$\mathcal{GH}([X,q],[Y,r]) \le \tfrac{1}{2}\Delta_R(q,r).$$

If q_0, q_1 are continuous pseudometrics on X and r_0, r_1 are continuous pseudometrics on Y, then

$$\big|\|q_0 - q_1\|_\infty - \|r_0 - r_1\|_\infty\big| \le \Delta_R(q_0,r_0) + \Delta_R(q_1,r_1),$$
$$\Delta_R((1-t)q_0 + tq_1, (1-t)r_0 + tr_1) \le (1-t)\Delta_R(q_0,r_0) + t\Delta_R(q_1,r_1)$$

for $0 \le t \le 1$. In particular, on a common compact space,

$$\mathcal{GH}([X,q_0],[X,q_1]) \le \tfrac{1}{2}\|q_0 - q_1\|_\infty. \tag{2.2}$$

Proof. Let $\pi_X \colon X \to X/{\sim_q}$ and $\pi_Y \colon Y \to Y/{\sim_r}$ be the quotient maps. The set

$$\{(\pi_X(x), \pi_Y(y)) \mid (x, y) \in R\}$$

is a correspondence because both quotient maps and both coordinate projections of R are surjective. For two of its elements, the difference of the quotient distances is

$$|q(x_0, x_1) - r(y_0, y_1)| \leq \Delta_R(q, r).$$

The correspondence formula in Theorem 2.2 proves the first bound.

For the uniform-norm estimate, fix $(x_0, y_0), (x_1, y_1) \in R$. The triangle inequality on the real line implies

$$\begin{aligned} |q_0(x_0, x_1) - q_1(x_0, x_1)| \leq{} & |r_0(y_0, y_1) - r_1(y_0, y_1)| \\ & + \Delta_R(q_0, r_0) + \Delta_R(q_1, r_1). \end{aligned}$$

Every pair in $X \times X$ has corresponding points in $Y \times Y$. Taking the supremum therefore bounds $\|q_0 - q_1\|_\infty$ by $\|r_0 - r_1\|_\infty$ plus $\Delta_R(q_0, r_0) + \Delta_R(q_1, r_1)$. Interchanging the carriers proves the reverse bound, and hence the absolute-value estimate in the statement.

For interpolation, the difference at the same two corresponding pairs is

$$\begin{aligned} &(1 - t)\big(q_0(x_0, x_1) - r_0(y_0, y_1)\big) \\ &\qquad + t\big(q_1(x_0, x_1) - r_1(y_0, y_1)\big). \end{aligned}$$

Its absolute value is bounded by the weighted sum of the two errors, since both weights are nonnegative. Taking the supremum proves the interpolation estimate. Finally, on a common carrier use $R = \{(x, x) \mid x \in X\}$. Its error is exactly $\|q_0 - q_1\|_\infty$, which proves (2.2). □

The following theorem allows us to refine open covers by balls of a continuous pseudometric.

Theorem 2.10 ([24, Theorem 2.1])**.** *Let X be a paracompact Hausdorff space, and let $\mathcal{U}$ be an open cover of X. Then there exist a continuous pseudometric $\mathcal{R} \colon X \times X \to [0, \infty)$ and a number $r > 0$ such that the family $\{B(x, r; \mathcal{R})\}_{x \in X}$ refines $\mathcal{U}$.*

2.3. **Probability measures and weak convergence.** We collect the measure-theoretic tools used for averaging probabilities and comparing distance operators on varying compact spaces.

For a metrizable space S, let $\mathcal{B}(S)$ be its Borel σ-algebra. A finite Borel measure μ on S is called *Radon* if it is inner regular by compact sets, meaning that

$$\mu(A) = \sup\{\mu(K) \mid K \subset A \text{ is compact}\} \quad (A \in \mathcal{B}(S)).$$

The notation $\mathcal{P}\mathrm{rob}(S)$ denotes those Radon measures with total mass one. A *probability law* on S means an element of $\mathcal{P}\mathrm{rob}(S)$. For Radon probabilities

$\mu \in \mathcal{P}\mathrm{rob}(S)$ and $\nu \in \mathcal{P}\mathrm{rob}(T)$, the notation $\mu \otimes \nu$ denotes their product probability on $S \times T$. It is characterized by

$$(\mu \otimes \nu)(A \times F) = \mu(A)\nu(F) \quad (A \in \mathcal{B}(S),\ F \in \mathcal{B}(T)).$$

For a Radon probability $\mu \in \mathcal{P}\mathrm{rob}(S)$, its *support* is

$$\mathrm{supp}(\mu) = \{x \in S \mid \mu(O) > 0 \text{ for every open } O \subset S \text{ with } x \in O\}.$$

We say that μ has *full support* if $\mathrm{supp}(\mu) = S$. Equivalently,

$$\forall O \subset S \text{ open and nonempty} \quad \mu(O) > 0.$$

If $f\colon S \to T$ is continuous and $\mu \in \mathcal{P}\mathrm{rob}(S)$, its pushforward is the probability $f_*\mu$ defined by

$$(f_*\mu)(A) = \mu(f^{-1}(A)) \qquad (A \in \mathcal{B}(T)).$$

For a nonempty metrizable space S, we write

$$C_b(S) = \{\varphi\colon S \to \mathbb{R} \mid \varphi \text{ is continuous and bounded}\},$$
$$\|\varphi\|_\infty = \sup_{x \in S} |\varphi(x)|.$$

For any compact metric space X, we define

$$C(X) = C(X,\mathbb{R}) = \{\varphi\colon X \to \mathbb{R} \mid \varphi \text{ is continuous}\}, \qquad \|\varphi\|_\infty = \max_{x \in X} |\varphi(x)|.$$

This is a real Banach space. Replacing the scalar field by $\mathbb{C}$ defines the complex Banach space $C(X,\mathbb{C})$.

We equip $\mathcal{P}\mathrm{rob}(S)$ with the weak topology, the coarsest topology for which all maps

$$\mathcal{P}\mathrm{rob}(S) \to \mathbb{R}, \qquad \mu \longmapsto \int_S \varphi \, d\mu \quad (\varphi \in C_b(S))$$

are continuous. We write $\mu_j \to \mu$ weakly when

$$\int_S \varphi \, d\mu_j \longrightarrow \int_S \varphi \, d\mu \quad (\varphi \in C_b(S)).$$

On a compact space all continuous functions are bounded. For bounded complex continuous test functions the same convergence follows by applying the definition to their real and imaginary parts.

We construct averages of measures by specifying their integrals of continuous functions. The following representation theorem makes this procedure well defined.

Theorem 2.11 (Riesz–Markov–Kakutani, [9, Theorem 7.10.4 and Corollary 7.10.5])**.** *Let (X,d) be a nonempty compact metric space. Let*

$$L\colon C(X,\mathbb{R}) \to \mathbb{R}$$

be a linear functional satisfying

$$L(\varphi) \geq 0 \quad \textit{whenever } \varphi \geq 0, \qquad L(1) = 1,$$

where 1 *denotes the constant function with value one. Then there is a unique Radon probability* $\mu \in \mathcal{P}\mathrm{rob}(X)$ *such that*

$$L(\varphi) = \int_X \varphi \, d\mu \quad (\varphi \in C(X, \mathbb{R})).$$

Positivity and normalization imply $|L(\varphi)| \leq \|\varphi\|_\infty$, so the functional L is automatically continuous. The cited representation theorem applies, and evaluating at the constant function 1 proves that the representing measure μ has total mass one.

Theorem 2.12 (Portmanteau, [9, Theorem 8.2.3, Corollary 8.2.4(a)])**.** *Let* S *be a metrizable space and let* $\mu, \mu_j \in \mathcal{P}\mathrm{rob}(S)$ *for* $j \in \mathbb{Z}_{\geq 0}$*. Then the following conditions are equivalent.*

(P1) *For every* $\varphi \in C_b(S)$*,*

$$\lim_{j \to \infty} \int_S \varphi \, d\mu_j = \int_S \varphi \, d\mu.$$

(P2) *For every closed subset* $F \subset S$*,*

$$\limsup_{j \to \infty} \mu_j(F) \leq \mu(F).$$

(P3) *For every open subset* $O \subset S$*,*

$$\mu(O) \leq \liminf_{j \to \infty} \mu_j(O).$$

We shall also use the following consequence for measures on varying compact sets.

Lemma 2.13. *Let* S_j, S *be nonempty compact subsets of a compact metric space* (Z, h) *with* $\mathsf{H}[h](S_j, S) \to 0$*. If* $\mu_j \to \mu$ *weakly in* $\mathcal{P}\mathrm{rob}(Z)$ *and* $\mu_j(S_j) = 1$*, then* $\mu(S) = 1$*.*

Proof. For every $r > 0$, the open set $O_r = \{z \in Z \mid \mathrm{dist}_h(z, S) > r\}$ is disjoint from S_j for all sufficiently large j. The Portmanteau theorem (Theorem 2.12) implies $\mu(O_r) \leq \liminf_j \mu_j(O_r) = 0$. Taking the union over $r = 1, 1/2, 1/3, \ldots$ proves the assertion. □

For probabilities on a compact metric space (Z, h), we use the Prokhorov distance

$$\mathsf{P}[h](\mu, \nu) = \inf \left\{ \eta > 0 \;\middle|\; \begin{array}{c} \mu(A) \leq \nu(A^\eta) + \eta, \\ \nu(A) \leq \mu(A^\eta) + \eta \quad \text{for every Borel } A \subset Z \end{array} \right\},$$

where $A^\eta = \{x \in Z \mid \mathrm{dist}_h(x, A) < \eta\}$. This is a metric on $\mathcal{P}\mathrm{rob}(Z)$ and induces weak convergence [33, author's manuscript, Definition 1.12, Proposition 1.13 and Lemma 1.15]. Prokhorov distance is unchanged when both measures are pushed forward by the same isometric embedding. Every Borel probability on a Polish space is Radon [9, Theorem 7.1.7]. The next facts therefore apply to our convention for probabilities.

A family $\mathcal{Q} \subset \mathcal{P}\mathrm{rob}(S)$ is called *uniformly tight* if for every $\varepsilon > 0$ there is a compact subset $K \subset S$ such that

$$\sup_{\mu \in \mathcal{Q}} \mu(S \setminus K) < \varepsilon.$$

Theorem 2.14 ([9, Theorems 8.9.3 and 8.9.4])**.** *Let S be a separable metrizable space. Then $\mathcal{P}\mathrm{rob}(S)$ is separable and metrizable in the weak topology. If S is Polish, then $\mathcal{P}\mathrm{rob}(S)$ is Polish. If S is compact, then $\mathcal{P}\mathrm{rob}(S)$ is compact.*

Theorem 2.15 (Prokhorov, [9, Theorem 8.6.2])**.** *Let S be a Polish space and let $\mathcal{Q} \subset \mathcal{P}\mathrm{rob}(S)$. Then $\mathcal{Q}$ has compact closure in the weak topology if and only if it is uniformly tight.*

In particular, every weakly convergent sequence of probabilities together with its limit is uniformly tight. If S is compact, then the whole space $\mathcal{P}\mathrm{rob}(S)$ is uniformly tight by taking $K = S$. Thus Prokhorov's theorem (Theorem 2.15) also explains the compactness assertion in Theorem 2.14.

Weak convergence is uniform over compact families of continuous test functions.

Lemma 2.16. *Let Z be a nonempty compact metric space, let $\mu_j \to \mu$ weakly in $\mathcal{P}\mathrm{rob}(Z)$, and let $\mathcal{F} \subset C(Z)$ be nonempty and compact in the uniform norm. Then*

$$\sup_{f \in \mathcal{F}} \left| \int_Z f \, d\mu_j - \int_Z f \, d\mu \right| \to 0.$$

The same conclusion holds for complex continuous functions.

Proof. For each j, take $f_j \in \mathcal{F}$ where the displayed supremum is attained. By compactness of $\mathcal{F}$, every subsequence has a further subsequence for which $f_j \to f \in \mathcal{F}$ uniformly. Along this further subsequence,

$$\left| \int_Z f_j \, d(\mu_j - \mu) \right| \leq 2\|f_j - f\|_\infty + \left| \int_Z f \, d(\mu_j - \mu) \right| \longrightarrow 0.$$

Thus the whole sequence of suprema tends to zero. The same argument applies to complex functions, since weak convergence controls both their real and imaginary parts. $\square$

The same estimate permits a continuous integrand to vary with compact parameters.

Lemma 2.17. *Let K and Z be nonempty compact metric spaces, let $\mu_j \to \mu$ weakly in $\mathcal{P}\mathrm{rob}(Z)$, and let continuous real or complex functions $F_j, F \colon K \times Z \to \mathbb{C}$ satisfy $\|F_j - F\|_\infty \to 0$. Then*

$$\sup_{k \in K} \left| \int_Z F_j(k, z) \, d\mu_j(z) - \int_Z F(k, z) \, d\mu(z) \right| \to 0.$$

Proof. Replacing F_j by F in the first integral changes it by at most $\|F_j - F\|_\infty$. The family $\{F(k,\cdot) \mid k \in K\}$ is compact in the uniform norm, so Lemma 2.16 controls the remaining difference. $\square$

For comparisons with metric measure geometry, we also recall the box distance.

Definition 2.18. An *mm-space* is a complete separable metric space (X, d_X) with a Borel probability measure μ_X. Two mm-spaces are *mm-isomorphic* if there is a measure-preserving isometry between their supports. Let $\mathcal{X}$ denote the set of these classes. Put $I = [0,1)$ and let $\mathcal{L}^1$ be Lebesgue measure on this interval. A *parameter* of X is a Borel map $\varphi\colon I \to X$ with $(\varphi)_*\mathcal{L}^1 = \mu_X$. Every mm-space has a parameter. The *box distance* $\square(X,Y)$ is the infimum of all $\varepsilon > 0$ for which there are parameters φ of X and ψ of Y and a Borel set $I_0 \subset I$ satisfying

$$\mathcal{L}^1(I_0) \geq 1 - \varepsilon, \qquad |d_X(\varphi(s), \varphi(t)) - d_Y(\psi(s), \psi(t))| \leq \varepsilon \quad (s, t \in I_0).$$

It is a metric on $\mathcal{X}$. The topology it induces is called the *box topology* [26, arXiv v1, Definitions 2.1, 2.6, 2.7 and Theorem 2.8].

2.4. **Contractions and absolute retracts.** We recall an extension theorem for continuous real functions and the terminology for contractions and absolute retracts.

Theorem 2.19 (Tietze–Urysohn, [15, Theorem 2.1.8])**.** *Let S be a normal space, let $A \subset S$ be closed, and let $M \geq 0$. Then every continuous function $f\colon A \to [-M, M]$ has a continuous extension $F\colon S \to [-M, M]$.*

A continuous map $f\colon A \to T$ is *null-homotopic* if there are a point $p \in T$ and a continuous map $H\colon A \times [0,1] \to T$ such that

$$H(a,0) = f(a), \qquad H(a,1) = p \qquad (a \in A).$$

A nonempty space is *contractible* if its identity map is null-homotopic.

The following lemma records the contraction obtained by scaling distances. Write $* = (\{0\}, 0)$ for the one-point metric space.

Lemma 2.20. *For compact metric spaces (X, d) and (Y, e) and real numbers $a, b \geq 0$, we have*

$$\mathcal{GH}([X, ad], [Y, be]) \leq a\mathcal{GH}((X,d),(Y,e)) + \frac{|a-b|}{2}\operatorname{diam}(Y, e). \tag{2.3}$$

Consequently the map

$$H\colon \mathcal{M} \times [0,1] \to \mathcal{M}, \qquad H((X,d),t) = [X, (1-t)d]$$

is continuous and satisfies

$$H((X,d),0) = (X,d), \qquad H((X,d),1) = *, \qquad H(*,t) = *.$$

In particular, $\mathcal{M}$ is contractible.

Proof. For a correspondence $R \subset X \times Y$ and pairs $(x_0, y_0), (x_1, y_1) \in R$,

$$\begin{aligned}|ad(x_0, x_1) - be(y_0, y_1)| &\le a|d(x_0, x_1) - e(y_0, y_1)| + |a - b|e(y_0, y_1) \\ &\le a \operatorname{dis}(R) + |a - b| \operatorname{diam}(Y, e).\end{aligned}$$

Lemma 2.9 now proves (2.3), also when one of the scaling factors is zero. The estimate proves continuity of the map

$$\mathcal{M} \times [0, \infty) \to \mathcal{M}, \qquad ((X, d), a) \longmapsto [X, ad],$$

including at $a = 0$. The three identities for H follow directly from its definition. □

For balls in Gromov–Hausdorff space we use a separate symbol,

$$\mathfrak{B}((X, d), r) = \{(Y, e) \in \mathcal{M} \mid \mathcal{GH}((X, d), (Y, e)) < r\}.$$

The correspondence formula also implies $\mathcal{GH}((X, d), *) = \operatorname{diam}(X, d)/2$.

We use absolute extension in the full metrizable category.

Definition 2.21. A metrizable space T is an *absolute extensor*, or AE, if every continuous map $f\colon A \to T$ from a closed subset of a metrizable space S extends continuously over S. It is an *absolute neighborhood extensor*, or ANE, if such a map extends over some open neighborhood of A in S. An *absolute retract*, or AR, is a metrizable space which is a retract whenever it is embedded as a closed subspace of a metrizable space. Replacing the ambient space by an open neighborhood defines an *absolute neighborhood retract*, or ANR.

Theorem 2.22 ([14, Theorem 12.1])**.** *For a metrizable target in the category of metrizable spaces, the AE and AR properties are equivalent, and the ANE and ANR properties are equivalent.*

Definition 2.23. A space T is *locally contractible at $x \in T$* if for every open neighborhood U of x there is an open neighborhood $V \subset U$ of x such that the inclusion $V \to U$ is null-homotopic.

The constant value of the homotopy can be required to be x. Indeed, the homotopy traces a path from x to its constant value, and its reversed path can be appended to the homotopy path of every point of V. For $\mathcal{M}$, this definition is equivalent to requiring that for every $(X, d) \in \mathcal{M}$ and $\varepsilon > 0$ there are $0 < \delta < \varepsilon$ and a continuous map

$$H\colon \mathfrak{B}((X, d), \delta) \times [0, 1] \to \mathfrak{B}((X, d), \varepsilon)$$

such that $H((Y, e), 0) = (Y, e)$ and $H((Y, e), 1) = (X, d)$.

Part I. The topology of Gromov–Hausdorff space I: Continuous invariant measure selection

Abstract. We construct an assignment of a full-support Borel probability measure to every nonempty compact metric space. The assignment respects all isometries and is continuous for simultaneous Hausdorff convergence of the underlying spaces and weak convergence of the measures in a common compact ambient space. We prove that the Gromov–Hausdorff–Prokhorov space of invariant full-support probabilities is Polish and that its projection onto the Gromov–Hausdorff space is an open surjection. A continuous selection of probability laws on the fibers, followed by averaging, yields the required assignment.

Keywords. Gromov–Hausdorff space, Gromov–Hausdorff–Prokhorov distance, invariant probability measure, full support, continuous selection.

2020 Mathematics Subject Classification. Primary 54E35; Secondary 54C65, 60B05, 60B10.

3. Introduction to Part I

Integrating functions on a compact metric space requires a choice of measure. When the space itself varies in the Gromov–Hausdorff topology, this choice must also vary continuously if the resulting integrals are to be compared. We study this selection problem for all nonempty compact metric spaces. Let $\mathcal{M}$ denote their isometry classes with the Gromov–Hausdorff distance $\mathcal{GH}$.

For metric measure spaces, Kazukawa, Nakajima, and Shioya [26, arXiv v1, Theorems 1.4–1.6] prove contractibility and local path connectedness for both the box and concentration topologies.

The selection problem here concerns the ordinary Gromov–Hausdorff topology on the underlying compact spaces and weak convergence of measures in common ambient spaces. Our construction retains the full underlying compact space throughout.

The main result is Theorem 5.6. It assigns to every nonempty compact metric space (X, d) a Borel probability measure $\mathfrak{m}_{X,d}$ with full support. For every isometry $u\colon (X, d) \to (Y, e)$, the assignment satisfies

$$\operatorname{supp}(\mathfrak{m}_{X,d}) = X, \qquad u_*\mathfrak{m}_{X,d} = \mathfrak{m}_{Y,e}.$$

Its continuity has the following concrete meaning. Whenever compact subspaces X_j and X of a compact metric space (Z, h) converge in Hausdorff distance, the selected measures, regarded as probabilities on Z, converge weakly. Here the metrics on the subspaces are the restrictions of h, and weak convergence means convergence of integrals of continuous real functions on Z.

For a fixed compact space, averaging over its compact isometry group produces invariant probabilities. The selection problem concerns compatibility of these choices as the space varies. Our proof uses the Gromov–Hausdorff–Prokhorov space of compact spaces with probability measures, retaining the whole underlying space even when a measure has smaller support. We prove that the subspace of invariant full-support measures is Polish and that its projection onto $\mathcal{M}$ is an open surjection. By Valov's theorem (Theorem 4.5), we obtain a continuous assignment of a probability law to each fiber. Averaging these laws produces the required measures. Using the Riesz–Markov–Kakutani theorem (Theorem 2.11), we define the averages. We prove their continuity by comparing integrals in a common compact ambient space.

As an application, we construct a topological embedding s of $\mathcal{M}$ into the Gromov–Hausdorff–Prokhorov space by attaching the selected measure. The map that forgets the measure is a continuous left inverse of s. We prove that the image $s(\mathcal{M})$ is a retract of the Gromov–Hausdorff–Prokhorov space and is contained in the subspace of invariant full-support probabilities (Corollary 5.7).

Dependence on the other parts. This part uses the common preliminaries in Section 2 and the cited literature, but no result from Parts II, III, or IV. In Part II, we use the selected measures from Theorem 5.6 to define distance operators and construct finite-dimensional approximations. The embedding and retraction in Corollary 5.7 are proved here directly from that selection theorem.

Organization. Section 4 contains the preliminaries for this part. In Subsection 4.1, we use the probability background from Subsection 2.3 to introduce the Gromov–Hausdorff–Prokhorov space and prove common ambient realization results. Subsection 4.2 recalls Haar probability and Valov's theorem. In Subsection 5.1, we prove that the invariant full-support subspace is Polish and that its projection onto the unmeasured Gromov–Hausdorff space is open. In Subsection 5.2, we construct continuous probability laws on the fibers and average them to prove Theorem 5.6. In Subsection 5.3, we identify the selected measured spaces as a retract of the Gromov–Hausdorff–Prokhorov space.

Conventions and notation. All compact metric spaces are nonempty. We identify a metric space with its isometry class when it occurs in $\mathcal{M}$. An isometry is a surjective isometric embedding. The notation $\mathcal{P}\mathrm{rob}(S)$ denotes the Radon probability measures on a metrizable space S, with weak convergence against bounded continuous real functions. Subscripts in $\mathfrak{m}_{X,d}$ refer to the specified metric space (X, d). For isometrically embedded subsets of a common ambient space, we use the same letters for the subsets and their restricted metrics. Sequences are indexed by $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$.

4. Preliminaries

We first introduce the Gromov–Hausdorff–Prokhorov space used in the selection construction. We then recall Haar probability and Valov's theorem on continuous laws. The notation for probabilities and weak convergence is fixed in Subsection 2.3.

4.1. **The Gromov–Hausdorff–Prokhorov space.** Using the probability measures and weak topology recalled in Subsection 2.3, we put all measured compact spaces in one metric space. At this stage a measure need not have full support.

Definition 4.1. Let $\mathcal{E}$ consist of isomorphism classes of triples (X, d, μ), where (X, d) is a nonempty compact metric space and $\mu \in \mathcal{P}\mathrm{rob}(X)$. An *isomorphism* from (X, d, μ) to (Y, e, ν) is an isometry $u\colon (X, d) \to (Y, e)$ such that $u_*\mu = \nu$. Define the Gromov–Hausdorff–Prokhorov distance by

$$\begin{aligned}&\mathcal{GHP}((X, d, \mu), (Y, e, \nu))\\ &\qquad = \inf_{Z,\iota,\kappa} \max\{\mathsf{H}[h](\iota(X), \kappa(Y)), \mathsf{P}[h](\iota_*\mu, \kappa_*\nu)\},\end{aligned} \tag{4.1}$$

where $\iota\colon (X, d) \to (Z, h)$ and $\kappa\colon (Y, e) \to (Z, h)$ range over isometric embeddings into compact metric spaces. We use the maximum convention for this unpointed distance. These conventions agree with [27, arXiv v5, Example 2.1(iii), Definition 2.4 and equation (8)], with probability measures as the additional structure. Let $\mathcal{I} \subset \mathcal{E}$ consist of the classes satisfying

$$\mathrm{supp}(\mu) = X, \qquad g_*\mu = \mu \quad \text{for every } g \in \mathrm{Isom}(X, d). \tag{4.2}$$

For full-support measures, the isomorphism in Definition 4.1 agrees with the usual mm-isomorphism in [26, arXiv v1, Definition 2.1]. That definition compares the supports of the measures. Our definition compares the entire compact spaces even when the measures have smaller supports. Accordingly, the Hausdorff distance term in (4.1) retains points outside the support.

Theorem 4.2 ([27, arXiv v5, Theorems 2.6 and 2.12 and Example 2.1(iii)])**.** *The function $\mathcal{GHP}$ is a complete separable metric on $\mathcal{E}$.*

The same framework realizes a convergent sequence in one compact ambient space.

Lemma 4.3. *If*

$$\mathcal{GHP}((X_j, d_j, \mu_j), (X, d, \mu)) \longrightarrow 0,$$

then there are a compact metric space (Z, h) and isometric embeddings

$$\iota_j\colon (X_j, d_j) \to (Z, h), \qquad \iota\colon (X, d) \to (Z, h)$$

such that

$$\mathsf{H}[h](\iota_j(X_j), \iota(X)) \longrightarrow 0, \qquad (\iota_j)_*\mu_j \longrightarrow \iota_*\mu \quad \textit{weakly in } \mathcal{P}\mathrm{rob}(Z). \tag{4.3}$$

Proof. By [27, arXiv v5, Lemma 2.5 and Example 2.1(iii)], we obtain a common compact ambient space in which the carriers converge in Hausdorff distance and the pushed-forward probabilities converge in Prokhorov distance. On a compact metric space, Prokhorov convergence is equivalent to weak convergence [33, author's manuscript, Lemma 1.15]. These are exactly the two conclusions in (4.3). □

4.2. **Haar probability and continuous laws.** We use Haar probability to average over the isometry group.

Theorem 4.4 (Haar probability, [13, Theorem 5.14 and Chapter 5, Section 3])**.** *Every compact metrizable topological group G admits a unique left invariant Radon probability $h \in \mathcal{P}\mathrm{rob}(G)$. Thus*

$$h(G) = 1, \qquad h(gA) = h(A) \quad (g \in G,\ A \in \mathcal{B}(G)).$$

This probability is also right invariant and has full support.

We apply the following consequence of Valov's theorem [36, arXiv v2, Theorem 1.1]. Its probabilities are Radon probabilities, with no compact-support restriction.

Theorem 4.5 (Valov, [36, arXiv v2, Theorem 1.1])**.** *Let $p\colon E \to T$ be an open continuous surjection between completely metrizable spaces. Then there is a continuous map $\Sigma\colon T \to \mathcal{P}\mathrm{rob}(E)$ such that $p_*\Sigma(t) = \delta_t$ for every $t \in T$.*

Here δ_t is the Dirac probability at t. The cited theorem constructs a continuous right inverse of the induced map on probabilities. Composing it with $t \mapsto \delta_t$ proves this formulation.

5. Continuous invariant probability measures

The distance operator used in Part II integrates functions over each compact metric space. Its construction therefore requires a probability measure on that space. We shall choose these measures so that isometries preserve them, every nonempty open set has positive measure, and convergence of the metric spaces implies weak convergence of the measures. The last requirement concerns measures on different spaces. We compare them after isometric embeddings into one compact metric space.

The construction has three steps. We first prove that the invariant full-support subspace $\mathcal{I}$ is Polish. We then prove that forgetting the measure is an open map. Finally, Valov's theorem (Theorem 4.5) provides a continuous probability law on each fiber, and averaging the measures represented by that law produces the required measure.

5.1. **The open projection to unmeasured spaces.** We next restrict the measured space $\mathcal{E}$ to invariant full-support measures and prove that forgetting the measure is an open map. Uniform control of integrals will describe the full-support condition, while limits of isometries will control invariance and the measures used to lift a convergent sequence of compact spaces. The required uniform estimates are in Lemmas 2.16 and 2.17.

The next lemma controls isometries when the compact spaces vary. The averaging argument below starts from the full isometry group, before an invariant measure has been constructed.

Lemma 5.1. *Let $X_j, X \subset Z$ be nonempty compact subsets of a compact metric space (Z, h) with $\mathsf{H}[h](X_j, X) \to 0$. Equip the product $Z \times Z$ with the* *maximum product metric. For any isometries* $g_j\colon X_j \to X_j$*, a subsequence of their graphs converges in* $\mathcal{K}(Z\times Z)$ *to the graph of an isometry* $g\colon X\to X$*. Along that subsequence, let* $\lambda_j \in \mathcal{P}\mathrm{rob}(X_j)$ *and* $\lambda\in\mathcal{P}\mathrm{rob}(X)$ *satisfy* $\lambda_j\to\lambda$ *weakly as probabilities on* Z*. Then*

$$(g_j)_*\lambda_j \longrightarrow g_*\lambda \quad \textit{weakly in } \mathcal{P}\mathrm{rob}(Z).$$

All subsets have their restricted metrics.

Proof. We first identify the limit of the graphs and then the limit of the probabilities supported on them. Write

$$h_\times((x_0,y_0),(x_1,y_1)) = \max\{h(x_0,x_1), h(y_0,y_1)\}.$$

For each j, put

$$R_j = \{(x, g_j(x)) \mid x\in X_j\}.$$

Compactness of $\mathcal{K}(Z\times Z)$ provides a subsequence, which we relabel, and a nonempty compact set $R\subset Z\times Z$ such that

$$\mathsf{H}[h_\times](R_j,R)\longrightarrow 0.$$

Step 1. The limiting graph. For $(x,y)\in R$, choose $(x_j,y_j)\in R_j$ converging to (x,y). Both coordinates belong to X_j, so

$$\begin{aligned}\mathrm{dist}_h(x,X) &\le h(x,x_j) + \mathsf{H}[h](X_j,X)\longrightarrow 0,\\ \mathrm{dist}_h(y,X) &\le h(y,y_j) + \mathsf{H}[h](X_j,X)\longrightarrow 0.\end{aligned}$$

Since X is closed, $R\subset X\times X$. Let $\mathrm{pr}_1,\mathrm{pr}_2\colon Z\times Z\to Z$ be the coordinate projections. They are 1-Lipschitz for $h_\times$, and $\mathrm{pr}_i(R_j) = X_j$ for $i=1,2$, because each g_j is surjective. Consequently,

$$\mathsf{H}[h](\mathrm{pr}_i(R),X)\le \mathsf{H}[h_\times](R,R_j) + \mathsf{H}[h](X_j,X)\longrightarrow 0.$$

Thus $\mathrm{pr}_1(R)=\mathrm{pr}_2(R)=X$.

For $(x_0,y_0),(x_1,y_1)\in R$, choose approximating pairs $(x_{i,j},y_{i,j})\in R_j$ for $i=0,1$. Then

$$\begin{aligned}h(x_0,x_1) &= \lim_j h(x_{0,j},x_{1,j})\\ &= \lim_j h(g_j(x_{0,j}), g_j(x_{1,j}))\\ &= \lim_j h(y_{0,j},y_{1,j}) = h(y_0,y_1).\end{aligned}$$

In particular, if $(x,y_0),(x,y_1)\in R$, then $h(y_0,y_1)=0$. Hence $y_0=y_1$. The first projection is surjective, so there is a unique map $g\colon X\to X$ with

$$R=\{(x,g(x))\mid x\in X\}.$$

The second projection being surjective implies $g(X)=X$, and the preceding distance identity implies

$$h(g(x_0),g(x_1)) = h(x_0,x_1)\quad (x_0,x_1\in X).$$

Hence g is an isometry.

Step 2. Probabilities on the graphs. Define

$$\mathsf{g}_j \colon X_j \to Z \times Z, \quad \mathsf{g}_j(x) = (x, g_j(x)),$$
$$\mathsf{g} \colon X \to Z \times Z, \quad \mathsf{g}(x) = (x, g(x)),$$

and put

$$\eta_j = (\mathsf{g}_j)_* \lambda_j.$$

Take any weakly convergent subsequence of η_j and denote its limit by η. Such subsequences exist by compactness of $\mathcal{P}\mathrm{rob}(Z \times Z)$. Since $\eta_j(R_j) = 1$, Lemma 2.13 and continuity of the first projection pr_1 imply

$$\eta(R) = 1, \qquad (\mathrm{pr}_1)_* \eta = \lim_j (\mathrm{pr}_1)_* \eta_j = \lim_j \lambda_j = \lambda,$$

where the limits are taken along the chosen subsequence. On R we have $\mathsf{g} \circ \mathrm{pr}_1 = \mathrm{id}_R$. Regarding η as a probability on R, we therefore obtain

$$\eta = (\mathsf{g} \circ \mathrm{pr}_1)_* \eta = \mathsf{g}_* \big((\mathrm{pr}_1)_* \eta \big) = \mathsf{g}_* \lambda.$$

Every weakly convergent subsequence has this same limit. Compactness of the probability space $\mathcal{P}\mathrm{rob}(Z \times Z)$ consequently implies

$$\eta_j \longrightarrow \mathsf{g}_* \lambda \quad \text{weakly in } \mathcal{P}\mathrm{rob}(Z \times Z).$$

Applying the continuous second projection pr_2 yields

$$(g_j)_* \lambda_j = (\mathrm{pr}_2)_* \eta_j \longrightarrow (\mathrm{pr}_2)_* (\mathsf{g}_* \lambda) = g_* \lambda.$$

This is the required weak convergence on Z. □

Proposition 5.2. *The space $\mathcal{I}$ is Polish.*

Proof. We first express the two conditions defining $\mathcal{I}$ as a countable intersection of open conditions. For $(X, d) \in \mathcal{M}$, $\mu \in \mathcal{P}\mathrm{rob}(X)$, and $r > 0$, we define the maps

$$b_r \colon X \times X \to [0, 1], \quad \alpha_r \colon X \to [0, 1], \quad m_r \colon \mathcal{E} \to [0, 1]$$

by

$$b_r(x, y) = \max\{1 - d(x, y)/r, 0\},$$
$$\alpha_r(x) = \int_X b_r(x, y)\, d\mu(y),$$
$$m_r(X, d, \mu) = \min_{x \in X} \alpha_r(x).$$

The function b_r is continuous on $X \times X$. Moreover, $|b_r(x, y) - b_r(x_1, y)| \leq d(x, x_1)/r$, so α_r is continuous and the minimum exists. For a convergent sequence in $\mathcal{E}$, use the compact embeddings from Lemma 4.3. The same formula defines b_r on $Z \times Z$, and $\{b_r(z, \cdot) \mid z \in Z\}$ is compact in $C(Z)$. By Lemma 2.17 with parameter space Z, $\alpha_{r,j} \to \alpha_r$ uniformly on Z. In particular,

$$|\min_{X_j} \alpha_{r,j} - \min_X \alpha_r| \leq \|\alpha_{r,j} - \alpha_r\|_\infty + \mathsf{H}[h](X_j, X)/r \longrightarrow 0.$$

Thus m_r is continuous on $\mathcal{E}$.

If $\operatorname{supp}(\mu) = X$, then for every $x \in X$,

$$\alpha_r(x) \geq \tfrac{1}{2}\mu(B(x, r/2; d)) > 0.$$

Continuity and compactness imply $m_r > 0$. Conversely, if $x \notin \operatorname{supp}(\mu)$, choose $k \in \mathbb{Z}_{\geq 0}$ with $\mu(B(x, 2^{-k}; d)) = 0$. Then $\alpha_{2^{-k}}(x) = 0$. We have proved

$$\operatorname{supp}(\mu) = X \quad \Longleftrightarrow \quad m_{2^{-k}}(X, d, \mu) > 0 \quad \text{for every } k \in \mathbb{Z}_{\geq 0}.$$

Define the subsets

$$\begin{aligned} U_k &= \{\xi \in \mathcal{E} \mid m_{2^{-k}}(\xi) > 0\}, \\ F_k &= \big\{\xi = (X, d, \mu) \in \mathcal{E} \;\big|\; \text{there is } g \in \operatorname{Isom}(X, d) \\ &\qquad\qquad \text{such that } \mathsf{P}[d](\mu, g_*\mu) \geq 2^{-k}\big\}. \end{aligned}$$

Each U_k is open. We next prove that F_k is closed. Take a sequence

$$\xi_j = (X_j, d_j, \mu_j) \in F_k$$

converging to $\xi = (X, d, \mu) \in \mathcal{E}$. For each j, choose $g_j \in \operatorname{Isom}(X_j, d_j)$ such that

$$\mathsf{P}[d_j](\mu_j, (g_j)_*\mu_j) \geq 2^{-k}.$$

By Lemma 4.3, realize this sequence and its limit in one compact metric space. By Lemma 5.1, we obtain a subsequence and an isometry $g \in \operatorname{Isom}(X, d)$ for which

$$(g_j)_*\mu_j \to g_*\mu \quad \text{weakly.}$$

Here and in the next limit, the index runs along this subsequence. Since Prokhorov distance induces weak convergence on that compact space, its triangle inequality implies

$$\mathsf{P}[d](\mu, g_*\mu) = \lim_j \mathsf{P}[d_j](\mu_j, (g_j)_*\mu_j) \geq 2^{-k}.$$

Thus $\xi \in F_k$, and hence F_k is closed. It follows that

$$\mathcal{I} = \bigcap_{k \in \mathbb{Z}_{\geq 0}} \big(U_k \cap (\mathcal{E} \setminus F_k)\big).$$

Thus $\mathcal{I}$ is a G_δ subset of the Polish space $\mathcal{E}$ by Theorem 4.2, and is Polish. □

Write

$$\mathsf{p}\colon \mathcal{I} \to \mathcal{M}, \qquad \mathsf{p}(X, d, \mu) = (X, d)$$

for the map that forgets the measure.

Lemma 5.3. *Every nonempty compact metric space (X, d) admits an invariant probability with full support.*

Proof. Fix a dense sequence $\{x_k\}_{k \in \mathbb{Z}_{\geq 0}}$ in X and put

$$\lambda = \sum_{k \in \mathbb{Z}_{\geq 0}} 2^{-(k+1)} \delta_{x_k}.$$

Since $\sum_{k\in\mathbb{Z}_{\geq 0}} 2^{-(k+1)} = 1$, this is a probability on X. For every nonempty open set $O \subset X$, we can take an index k with $x_k \in O$ because the sequence $\{x_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is dense in X. Then

$$\lambda(O) \geq \lambda(\{x_k\}) \geq 2^{-(k+1)} > 0.$$

Thus $\operatorname{supp}(\lambda) = X$.

We next verify compactness of $G = \operatorname{Isom}(X, d)$ for the uniform metric

$$d_\infty(f, g) = \sup_{x\in X} d(f(x), g(x)) \quad (f, g \in C(X, X)).$$

For every $g \in G$ we have

$$d(g(x), g(y)) = d(x, y) \quad (x, y \in X).$$

Hence G is equicontinuous. The combined image $\bigcup_{g\in G} g(X) = X$ is compact and hence totally bounded. The Arzelà–Ascoli theorem (Theorem 2.8) therefore implies that G has compact closure in $C(X, X)$.

To prove that the group G is closed, let $g_j \in G$ and $f \in C(X, X)$ satisfy $d_\infty(g_j, f) \to 0$. For all $x, y \in X$,

$$d(f(x), f(y)) = \lim_j d(g_j(x), g_j(y)) = d(x, y).$$

Thus $f\colon (X, d) \to (X, d)$ is an isometric embedding. Since X is compact, [11, Theorem 1.6.14] implies $f(X) = X$. Therefore, $f \in G$. This proves that G is closed and compact. The following estimates

$$\begin{aligned} d_\infty(g_j \circ h_j, g \circ h) &\leq d_\infty(g_j, g) + d_\infty(h_j, h), \\ d_\infty(g_j^{-1}, g^{-1}) &= d_\infty(g_j, g) \end{aligned}$$

for elements of G show that composition and inversion are continuous. Hence G is a compact metrizable topological group. By the Haar theorem (Theorem 4.4), we obtain a normalized left invariant probability h_G on G. Define its average by requiring that for every $\varphi \in C(X)$,

$$\int_X \varphi\, d\overline{\lambda} = \int_G \int_X \varphi(gx)\, d\lambda(x)\, dh_G(g).$$

By the Riesz–Markov–Kakutani theorem (Theorem 2.11), this positive normalized linear functional represents a unique probability on X. Left invariance of h_G implies $a_*\overline{\lambda} = \overline{\lambda}$ for every $a \in G$. For every nonempty open $O \subset X$, approximate its indicator from below by continuous functions taking values in $[0, 1]$. Monotone convergence in the defining identity then implies

$$\overline{\lambda}(O) = \int_G \lambda(g^{-1}O)\, dh_G(g) > 0,$$

since each integrand is positive. Hence $(X, d, \overline{\lambda}) \in \mathcal{I}$, as claimed. $\square$

Lemma 5.4. *Let $\xi = (X, d, \mu) \in \mathcal{I}$ and assume that $(X_j, d_j) \to (X, d)$ in $\mathcal{M}$. Then there are invariant probabilities ν_j with full support on X_j such that*

$$(X_j, d_j, \nu_j) \longrightarrow \xi \quad \textit{in } \mathcal{I}.$$

Proof. By Lemma 2.7, choose isometric embeddings into a compact (Z, h) and identify the spaces with their images. By the Hausdorff continuity of the probability-measure functor [27, arXiv v5, Lemma 2.13 and Example 2.1(iii)], there are probabilities $\mu_j \in \mathcal{P}\mathrm{rob}(X_j)$ such that

$$\mathsf{P}[h](\mu_j, \mu) \longrightarrow 0.$$

Thus $\mu_j \to \mu$ weakly in $\mathcal{P}\mathrm{rob}(Z)$. Choose a probability θ_j with full support on each X_j, for example by assigning positive weights summing to one to a dense sequence. For $0 < t_j < 1$ with $t_j \to 0$, put

$$\lambda_j = (1 - t_j)\mu_j + t_j\theta_j.$$

This probability has full support. For every real continuous function φ on Z,

$$\left| \int \varphi \, d\lambda_j - \int \varphi \, d\mu_j \right| \leq 2t_j \|\varphi\|_\infty \longrightarrow 0.$$

Hence $\lambda_j \to \mu$ weakly.

Average λ_j over $G_j = \mathrm{Isom}(X_j, d_j)$ with normalized Haar probability to obtain ν_j. The averaging construction in Lemma 5.3 shows that these probabilities are invariant and have full support. For each $\varphi \in C(Z)$, we claim that

$$\sup_{g \in G_j} \left| \int_Z \varphi \, d(g_*\lambda_j) - \int_Z \varphi \, d\mu \right| \longrightarrow 0. \tag{5.1}$$

For the sake of contradiction, suppose that there are $\varepsilon > 0$, a subsequence, and $g_j \in G_j$ such that

$$\left| \int_Z \varphi \, d((g_j)_*\lambda_j) - \int_Z \varphi \, d\mu \right| \geq \varepsilon$$

for every index in the subsequence. By Lemma 5.1, a further subsequence satisfies

$$(g_j)_*\lambda_j \to g_*\mu = \mu \quad \text{for some } g \in \mathrm{Isom}(X, d).$$

This contradicts that lower bound. Averaging (5.1) over G_j yields $\nu_j \to \mu$ weakly. Consequently,

$$\xi_j := (X_j, d_j, \nu_j) \longrightarrow \xi, \qquad \mathsf{p}(\xi_j) = (X_j, d_j).$$

□

Proposition 5.5. *The map*

$$\mathsf{p} \colon \mathcal{I} \to \mathcal{M}, \qquad \mathsf{p}(X, d, \mu) = (X, d),$$

is a continuous open surjection.

Proof. The inequality

$$\mathcal{GH}((X,d),(Y,e)) \le \mathcal{GHP}((X,d,\mu),(Y,e,\nu))$$

proves continuity of p. Lemma 5.3 proves surjectivity. To prove openness, let $O \subset \mathcal{I}$ be open and $\xi \in O$. For the sake of contradiction, suppose that $\mathsf{p}(O)$ contains no neighborhood of $\mathsf{p}(\xi)$. Then we can choose

$$(X_j, d_j) \in \mathfrak{B}(\mathsf{p}(\xi), 2^{-j}) \setminus \mathsf{p}(O).$$

By Lemma 5.4, we can take lifts ξ_j such that $\xi_j \to \xi$, so $\xi_j \in O$ for all sufficiently large j. This implies $(X_j, d_j) \in \mathsf{p}(O)$, a contradiction. Thus $\mathsf{p}(O)$ is open. □

5.2. **Continuous laws and their averages.** We now average the laws on invariant measures.

Using the Riesz–Markov–Kakutani theorem (Theorem 2.11), we define these averages. We prove their continuity as the compact metric space varies by comparing integrals in a common compact ambient space.

Theorem 5.6. *For every nonempty compact metric space (X,d), there is a probability $\mathfrak{m}_{X,d} \in \mathcal{P}\mathrm{rob}(X)$ with $\mathrm{supp}(\mathfrak{m}_{X,d}) = X$. The assignment can be chosen to satisfy the following properties. For every nonempty compact metric space (Y,e) and every isometry $u\colon (X,d) \to (Y,e)$, we have*

$$u_*\mathfrak{m}_{X,d} = \mathfrak{m}_{Y,e}.$$

Moreover, let (X_j,d_j) and (X,d) be nonempty compact metric spaces, and let $\iota_j\colon (X_j,d_j) \to (Z,h)$ and $\iota\colon (X,d) \to (Z,h)$ be isometric embeddings into any compact metric space. Then

$$\mathsf{H}[h](\iota_j(X_j), \iota(X)) \to 0 \quad \Longrightarrow \quad (\iota_j)_*\mathfrak{m}_{X_j,d_j} \to \iota_*\mathfrak{m}_{X,d} \quad \text{weakly in } \mathcal{P}\mathrm{rob}(Z).$$

Proof. By Propositions 5.2 and 5.5, the map $\mathsf{p}\colon \mathcal{I} \to \mathcal{M}$ is an open continuous surjection between Polish spaces. By Valov's theorem (Theorem 4.5), we obtain a continuous map $\Sigma\colon \mathcal{M} \to \mathcal{P}\mathrm{rob}(\mathcal{I})$ such that $\mathsf{p}_*\Sigma(X,d) = \delta_{(X,d)}$. For a fixed representative (X,d), write

$$\mathcal{L}_X = \mathsf{p}^{-1}((X,d)).$$

This is a closed Polish subspace of $\mathcal{I}$, and $\Sigma(X,d)(\mathcal{L}_X) = 1$. If $\xi = (Y,e,\nu) \in \mathcal{L}_X$, choose an isometry $u\colon (Y,e) \to (X,d)$. We denote the resulting measure on this fixed representative by

$$I_{X,\xi} = u_*\nu, \qquad I_{X,\xi}(A) = \nu(u^{-1}(A)) \quad (A \in \mathcal{B}(X)).$$

This notation defines a map $I_{X,\bullet}\colon \mathcal{L}_X \to \mathcal{P}\mathrm{rob}(X)$, $\xi \mapsto I_{X,\xi}$. Here the symbol $\bullet$ indicates the variable ξ in the map $I_{X,\bullet}$. If $v\colon (Y,e) \to (X,d)$ is another isometry, then $v^{-1}u \in \mathrm{Isom}(Y,e)$ and

$$u_*\nu = v_*\big((v^{-1}u)_*\nu\big) = v_*\nu.$$

The definition is therefore independent of the chosen isometry and of the representative of ξ.

We first prove continuity of this transport of measures. Identify the isometrically embedded spaces X_j, X with their images in Z. If $\xi_j \in \mathcal{L}_{X_j}$ and $\xi \in \mathcal{L}_X$ satisfy $\xi_j \to \xi$ in $\mathcal{I}$, then

$$I_{X_j,\xi_j} \longrightarrow I_{X,\xi} \quad \text{weakly in } \mathcal{P}\mathrm{rob}(Z). \tag{5.2}$$

Indeed, take any weakly convergent subsequence of I_{X_j,ξ_j} and denote its limit by ν. Such subsequences exist by compactness of $\mathcal{P}\mathrm{rob}(Z)$. Until this limit is identified, the index runs along the chosen subsequence. The limit is concentrated on X, by Lemma 2.13. The definition of $\mathcal{GHP}$ and weak convergence on Z then imply

$$\xi_j \longrightarrow (X, d, \nu).$$

Uniqueness of limits implies $(X, d, \nu) = \xi$. Thus $\nu = g_* I_{X,\xi}$ for some isometry $g\colon X \to X$. The measure $I_{X,\xi}$ is invariant, so $\nu = I_{X,\xi}$. To deduce convergence of the whole sequence $\{I_{X_j,\xi_j}\}_{j\in\mathbb{Z}_{\geq 0}}$, assume for the sake of contradiction that (5.2) fails. Then some weakly open neighborhood of $I_{X,\xi}$ is missed by a subsequence. Compactness of $\mathcal{P}\mathrm{rob}(Z)$ provides a further convergent subsequence. The preceding argument identifies its limit with $I_{X,\xi}$, so its terms eventually belong to that neighborhood, a contradiction. With the space X fixed, this also proves continuity of $I_{X,\bullet}$.

Define $\mathfrak{m}_{X,d}$ by requiring the following identity for every $\varphi \in C(X)$.

$$\int_X \varphi \, d\mathfrak{m}_{X,d} = \int_{\mathcal{L}_X} \left(\int_X \varphi \, dI_{X,\xi} \right) d\Sigma(X,d)(\xi). \tag{5.3}$$

Figure 1 records the two levels of probability measures.

$$\begin{tikzcd}
\Sigma(X,d) \in \mathcal{P}\mathrm{rob}(\mathcal{L}_X) \arrow[d, "\text{transport by } I_{X,\bullet}"] \\
(I_{X,\bullet})_*\Sigma(X,d) \in \mathcal{P}\mathrm{rob}(\mathcal{P}\mathrm{rob}(X)) \arrow[d, "\text{average}"] \\
\mathfrak{m}_{X,d} \in \mathcal{P}\mathrm{rob}(X)
\end{tikzcd}$$

FIGURE 1. For a fixed representative (X, d), the law $\Sigma(X, d)$ on the fiber $\mathcal{L}_X$ is transported to a law on $\mathcal{P}\mathrm{rob}(X)$. Its average is defined by (5.3).

The right-hand side is well defined because $I_{X,\bullet}$ is continuous and the inner integral has absolute value at most $\|\varphi\|_\infty$. As a function of the real continuous test function φ, it is a positive linear functional taking the constant function 1 to 1. By the Riesz–Markov–Kakutani theorem (Theorem 2.11), we obtain a unique probability $\mathfrak{m}_{X,d}$ with this identity. For $g \in \mathrm{Isom}(X, d)$

and $\varphi \in C(X)$, invariance of each measure in the fiber implies

$$\int_X \varphi \circ g \, dI_{X,\xi} = \int_X \varphi \, dI_{X,\xi}.$$

Integrating this equality against the law in (5.3) yields

$$\int_X \varphi \, d(g_* \mathfrak{m}_{X,d}) = \int_X \varphi \circ g \, d\mathfrak{m}_{X,d} = \int_X \varphi \, d\mathfrak{m}_{X,d}.$$

Uniqueness in the Riesz–Markov–Kakutani theorem (Theorem 2.11) proves invariance of the averaged measure. For a nonempty open $O \subset X$, choose $x \in O$ and $r > 0$ with $B(x, r; d) \subset O$. Define the continuous function $\varphi\colon X \to [0,1]$ by $\varphi(y) = \max\{1 - 2d(x,y)/r, 0\}$. Then φ is nonnegative, vanishes outside O, and is positive at x. Since each $I_{X,\xi}$ has full support,

$$\int_X \varphi \, dI_{X,\xi} > 0 \quad (\xi \in \mathcal{L}_X).$$

The positive inner integrals need not have a common positive lower bound. For each integer $n \geq 1$, put

$$A_n = \left\{ \xi \in \mathcal{L}_X \;\middle|\; \int_X \varphi \, dI_{X,\xi} \geq 1/n \right\}.$$

These are Borel sets by continuity of the inner integral, and their union is the whole fiber $\mathcal{L}_X$. The law $\Sigma(X,d)$ assigns mass one to $\mathcal{L}_X$, so at least one A_n has positive mass. Since $0 \leq \varphi \leq 1$ and the function vanishes outside O,

$$\mathfrak{m}_{X,d}(O) \geq \int_X \varphi \, d\mathfrak{m}_{X,d} \geq \frac{1}{n} \Sigma(X,d)(A_n) > 0.$$

This proves full support. Finally, for an isometry $u\colon (X,d) \to (Y,e)$, the fibers and their laws are the same and $I_{Y,\xi} = u_* I_{X,\xi}$. For every $\varphi \in C(Y)$, apply (5.3) on X to $\varphi \circ u$. The change-of-variables identity for each fiber measure then implies

$$\begin{aligned}
\int_Y \varphi \, d(u_* \mathfrak{m}_{X,d}) &= \int_{\mathcal{L}_X} \left(\int_X \varphi \circ u \, dI_{X,\xi} \right) d\Sigma(X,d)(\xi) \\
&= \int_{\mathcal{L}_Y} \left(\int_Y \varphi \, dI_{Y,\xi} \right) d\Sigma(Y,e)(\xi) \\
&= \int_Y \varphi \, d\mathfrak{m}_{Y,e}.
\end{aligned}$$

Uniqueness of the representing measure proves the required compatibility with isometries.

It remains to prove continuity as the underlying spaces vary. We use the prescribed embeddings into Z and identify each carrier with its image. Fix a real continuous function $\varphi \in C(Z)$. We will express its integral against each selected measure using one bounded continuous function on a fixed space.

Step 1. A closed domain for the fiber integrals. Put

$$T = \{0\} \cup \{2^{-j} \mid j \in \mathbb{Z}_{\geq 0}\}, \qquad \mathsf{X}_0 = X, \qquad \mathsf{X}_{2^{-j}} = X_j.$$

All carriers have the restricted ambient metric. The map from T to $\mathcal{M}$ that assigns the isometry class of X_t to t is continuous. Consequently,

$$A = \{(t, \xi) \in T \times \mathcal{I} \mid \mathsf{p}(\xi) = [\mathsf{X}_t]\}$$

is closed, since it is the equality set of two continuous maps to the metric space $\mathcal{M}$. Here the brackets denote the isometry class with the restricted metric. Define a function $F\colon A \to \mathbb{R}$ by

$$F(t, \xi) = \int_{\mathsf{X}_t} \varphi \, dI_{\mathsf{X}_t, \xi}.$$

Equation (5.2) proves continuity at parameter 0. At an isolated parameter, continuity follows from continuity on the fixed fiber proved above. Moreover, $|F| \le \|\varphi\|_\infty$.

Step 2. Extension and weak convergence of the laws. The product $T \times \mathcal{I}$ is metrizable and hence normal. By the Tietze–Urysohn theorem (Theorem 2.19), F extends to a continuous function $\widetilde{F}\colon T \times \mathcal{I} \to \mathbb{R}$ with $|\widetilde{F}| \le \|\varphi\|_\infty$. Write

$$\lambda_t = \Sigma([\mathsf{X}_t]).$$

Continuity of Σ implies $\lambda_{2^{-j}} \to \lambda_0$ weakly. It follows that

$$\delta_{2^{-j}} \otimes \lambda_{2^{-j}} \longrightarrow \delta_0 \otimes \lambda_0 \quad \text{weakly on } T \times \mathcal{I}.$$

To verify this convergence, let $\psi\colon T \times \mathcal{I} \to \mathbb{R}$ be bounded and continuous. Since $\lambda_{2^{-j}} \to \lambda_0$ weakly on the Polish space $\mathcal{I}$, Prokhorov's theorem (Theorem 2.15) implies that the family

$$\{\lambda_{2^{-j}} \mid j \in \mathbb{Z}_{\ge 0}\} \cup \{\lambda_0\}$$

is uniformly tight. For every $0 < \varepsilon < 1$, there is a nonempty compact set $K \subset \mathcal{I}$ such that

$$\sup_{t \in T} \lambda_t(\mathcal{I} \setminus K) < \varepsilon.$$

Uniform continuity on $T \times K$ and weak convergence of $\lambda_{2^{-j}}$ to λ_0 imply

$$\begin{aligned} &\left| \int \psi(2^{-j}, \xi) \, d\lambda_{2^{-j}}(\xi) - \int \psi(0, \xi) \, d\lambda_0(\xi) \right| \\ &\quad \le \sup_{\xi \in K} |\psi(2^{-j}, \xi) - \psi(0, \xi)| + 2\|\psi\|_\infty \varepsilon \\ &\qquad + \left| \int \psi(0, \xi) \, d(\lambda_{2^{-j}} - \lambda_0)(\xi) \right|. \end{aligned}$$

The first and last terms tend to zero. Taking the upper limit and then letting $\varepsilon \downarrow 0$ proves weak convergence of the product laws.

Step 3. Integrals of the averaged measures. Each product law in Step 2 is concentrated on A. By (5.3),

$$\int_Z \varphi \, d\mathfrak{m}_{X_j,d_j} = \int \widetilde{F} \, d(\delta_{2^{-j}} \otimes \lambda_{2^{-j}}) \longrightarrow \int \widetilde{F} \, d(\delta_0 \otimes \lambda_0) = \int_Z \varphi \, d\mathfrak{m}_{X,d}.$$

Since this holds for every real continuous φ on the compact ambient space Z, the selected measures $\mathfrak{m}_{X_j,d_j}$ converge weakly to $\mathfrak{m}_{X,d}$ as asserted. □

5.3. **A retract of the measured space.** The selected measures also relate the topology of the unmeasured and measured spaces directly. Theorem 5.6 provides a continuous section of the forgetful map from $\mathcal{E}$ to $\mathcal{M}$.

Corollary 5.7. *The map*

$$s\colon \mathcal{M} \to \mathcal{E}, \qquad s((X,d)) = (X, d, \mathfrak{m}_{X,d})$$

is a topological embedding. Its image is contained in $\mathcal{I}$ *and is a retract of* $\mathcal{E}$*. Consequently,* $\mathcal{M}$ *is homeomorphic to a retract of the Gromov–Hausdorff–Prokhorov space* $\mathcal{E}$*, also when the latter is restricted to full-support probabilities or to invariant full-support probabilities.*

Proof. Define the forgetful map on the whole measured space by

$$p\colon \mathcal{E} \to \mathcal{M}, \qquad p((X,d,\mu)) = (X,d).$$

Isomorphic measured spaces have isometric underlying spaces, so this map is well defined. By (4.1),

$$\mathcal{GH}\big(p((X,d,\mu)), p((Y,e,\nu))\big) \le \mathcal{GHP}((X,d,\mu),(Y,e,\nu)).$$

Thus p is continuous. The isometry compatibility and full-support assertions of Theorem 5.6 ensure that s is well defined and takes its values in $\mathcal{I}$. To check continuity, realize any GH convergent sequence in a common compact ambient space by Lemma 2.7. Theorem 5.6 implies weak convergence of the selected measures there. The Prokhorov metric metrizes weak convergence, so (4.1) implies convergence of the selected measured spaces. This proves continuity of s.

Since $p \circ s = \mathrm{id}_{\mathcal{M}}$, the map s is injective and its inverse on its image is the restriction of p. It is therefore a topological embedding. The continuous map

$$r = s \circ p\colon \mathcal{E} \to s(\mathcal{M})$$

fixes every point of $s(\mathcal{M})$, so it is a retraction. Its restrictions to the full-support subspace and to $\mathcal{I}$ are retractions onto the same image. □

Part II. The topology of Gromov–Hausdorff space II: Continuous finite-dimensional approximation

Abstract. We construct local finite-dimensional approximations of compact metric spaces that vary continuously in the Gromov–Hausdorff topology. Near every nonempty compact metric space and for every positive error tolerance, we obtain a fixed finite dimension, a varying norm, and continuous point maps whose induced pseudometrics approximate the original distances uniformly. The construction respects isometries and is well defined up to orthogonal changes of coordinates. Using continuous invariant full-support probabilities, we approximate distance functions in finite spectral subspaces of the distance operator by unique best approximations in finite L^p norms. Continuity of the spectral subspaces and the minimizers yields continuity of the local models.

Keywords. Gromov–Hausdorff space, finite-dimensional approximation, distance operator, spectral subspace, best approximation.

2020 Mathematics Subject Classification. Primary 54E35; Secondary 47A55, 47A58.

6. Introduction to Part II

A compact metric space can be represented by its distance functions in an infinite-dimensional function space. We study finite-dimensional approximations of this representation that remain continuous when the compact metric space varies in the Gromov–Hausdorff topology. Let $\mathcal{M}$ denote the space of isometry classes of nonempty compact metric spaces. Our main result is Theorem 8.8.

For one compact metric space, distances to the points of a finite net provide a finite-dimensional approximation. When the space varies, we also require the approximations to respect isometries and convergence. Our theorem constructs such approximations near each compact metric space. Their dimension is fixed on the neighborhood, and their errors in all pairwise distances have a common bound. Along a convergent sequence of spaces, the norms and point maps converge after suitable orthonormal bases have been chosen.

For every nonempty compact metric space (X, d) and every $b > 0$, the theorem constructs a neighborhood $\mathcal{W} \subset \mathcal{M}$ and a fixed integer $n \geq 1$. For each $(Y, e) \in \mathcal{W}$, there are a norm N_Y on $\mathbb{R}^n$ and a continuous map

$$\mathbf{v}_Y : Y \to \mathbb{R}^n.$$

The induced pseudometric and its uniform error are

$$q_Y(y, z) = N_Y(\mathbf{v}_Y(y) - \mathbf{v}_Y(z)), \qquad \mathcal{E}(Y, e) = \sup_{y,z \in Y} |q_Y(y, z) - e(y, z)|.$$

The function $\mathcal{E}$ is continuous on $\mathcal{W}$ and satisfies $\mathcal{E}(X, d) < b$. Thus, after shrinking $\mathcal{W}$, the same bound holds throughout the neighborhood. The norms and point maps vary continuously after suitable choices of orthonormal coordinates. Isometries of the compact metric spaces and changes of coordinates are accounted for by orthogonal transformations, as specified in Theorem 8.8. These are finite-dimensional representations of the points of each compact space, rather than coordinate charts on $\mathcal{M}$.

The theorem applies at every compact center and at every positive error scale. The dimension and the neighborhood may depend on both choices. Within that neighborhood, the image $\mathbf{v}_Y(Y)$ is a compact subset of one fixed vector space equipped with the varying norm N_Y. We also use N_Y to denote the metric $(v,w) \mapsto N_Y(v-w)$ on $\mathbb{R}^n$. The correspondence $\{(y, \mathbf{v}_Y(y)) \mid y \in Y\}$ has distortion at most $\mathcal{E}(Y,e)$. Consequently, Theorem 2.2 implies

$$\mathcal{GH}((Y,e),(\mathbf{v}_Y(Y),N_Y)) < \frac{b}{2}$$

after the neighborhood has been shrunk as above. Thus the point maps provide finite-dimensional metric approximations with simultaneous control of the error and of their variation. For each convergent sequence in the neighborhood, we can choose orthonormal bases so that the norms and point maps converge. The theorem does not require a continuous choice of bases on the whole neighborhood.

We use the continuous invariant full-support assignment of probabilities constructed in Theorem 5.6 of Part I. For (X,d), the associated distance operator $T_{X,d}\colon L^2(X,\mathfrak{m}_{X,d}) \to L^2(X,\mathfrak{m}_{X,d})$ is defined by

$$(T_{X,d}f)(x) = \int_X d(x,z)f(z)\, d\mathfrak{m}_{X,d}(z).$$

Maria, Oudot, and Solomon study this operator and its spectral coordinates [28, Section 3]. We use entire finite spectral subspaces and unique best approximations in finite L^p norms. Here $2 \le p < \infty$, and the norm is taken with respect to the selected probability measure. Keeping the whole subspace allows multiplicities of eigenvalues to be handled by orthogonal changes of basis.

The local models also have applications within this part. Corollary 8.10 approximates every compact continuous family uniformly by spaces admitting isometric embeddings into normed spaces of one bounded finite dimension. Combining local pseudometrics by a partition of unity yields homotopy density of the doubling compact metric spaces in $\mathcal{M}$ by Corollary 8.12. This complements the author's description of the set of all doubling compact metric spaces as a dense meager F_σ set in [20, Theorem 1.1]. The approximation preserves connectedness and path connectedness, and can also enforce uniform perfectness. Finally, Proposition 8.14 and example 8.15 show that the dimension must depend on accuracy for spaces of infinite Assouad dimension, even when the space is homeomorphic to a convergent sequence with its limit.

Part III applies these local models to the topology of $\mathcal{M}$. Their linear structure and orthogonal symmetry allow the point maps to be combined by a partition of unity after the varying norms are represented in a fixed Banach space. The interpolation result in [25, Theorem 1.1] controls the uniform error when extending prescribed metrics from a discrete family of closed subsets of a fixed metrizable space. Here we use Theorem 8.8 to construct maps and norms on varying compact spaces for the subsequent absolute retract argument.

Dependence on the other parts. In this part, we use the continuous invariant full-support probability measures constructed in Theorem 5.6 of Part I to define the distance operators and prove continuity of the finite-dimensional approximations. Beyond the common preliminaries, no other result of Part I is required. No result of Part III or Part IV is used. The local approximation theorem and its applications to homotopy density are proved in this part. In Part III, we use Theorem 8.8 to prove the absolute retract property of $\mathcal{M}$.

Organization. In Section 7, we recall best approximation in L^p, spectral decomposition and perturbation, and square roots of positive definite matrices. In Subsection 8.1, we define the distance operator and study its finite spectral subspaces. In Subsection 8.2, we prove approximation of distance functions using finite spectral subspaces and unique best approximations in finite L^p norms. In Subsection 8.3, we prove continuity of the finite spectral subspaces and construct compatible orthonormal bases along convergent sequences of spaces. In Subsection 8.4, we combine these results to construct the local maps and varying norms in Theorem 8.8, including the singleton case. In Subsection 8.5, we prove simultaneous approximation of compact families and homotopy density of doubling spaces. In Subsection 8.6, we explain the necessary dependence of the model dimension on the approximation error.

Conventions and notation. All compact metric spaces are nonempty, and finite-dimensional coordinate spaces are real. Weak convergence of probabilities and the uniform integral estimates are recalled in Subsection 2.3. Function spaces are real unless complex scalars are explicitly indicated for spectral perturbation. An isometry is a surjective isometric embedding. The pair (X, d) specifies both the underlying set and its metric. In particular, $\mathfrak{m}_{X,d}$ and $T_{X,d}$ depend on that pair. When compact spaces are embedded isometrically into a common ambient space, we use the same letters for the embedded subsets and the restricted metrics. The notation id_X denotes the identity map on X. We omit the subscript when its domain is specified. Sequences are indexed by $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

7. Preliminaries

We recall best approximation in L^p, the spectral theorem for compact self-adjoint operators, and the perturbation results used to compare finite spectral subspaces. The last subsection recalls a matrix theorem used to normalize bases of finite spectral subspaces.

7.1. **Best approximation in Lebesgue spaces.** Fix a nonempty compact metric space (X, d). For a probability μ on X and an exponent $1 \leq p < \infty$, the space $L^p(X, \mu)$ consists of measurable real functions with finite norm

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p},$$

where functions equal μ-almost everywhere are identified.

Theorem 7.1 (Best approximation, a consequence of [10, Lemma 3.2.36(b), Theorem 5.6.1])**.** *Let $1 < p < \infty$, let μ be a probability measure on X, and let $A \subset L^p(X, \mu)$ be nonempty, closed, and convex. Then for every $f \in L^p(X, \mu)$, there is a unique $g \in A$ such that*

$$\|f - g\|_p = \inf_{v \in A} \|f - v\|_p.$$

Proof. By the cited Theorem 5.6.1, $L^p(X, \mu)$ is uniformly convex. The translated set $\{v - f \mid v \in A\}$ is nonempty, closed, and convex. By Lemma 3.2.36(b) of the cited source, we obtain a unique element of minimum norm in this set. Translating that element by f proves the assertion. $\square$

7.2. **Compact operators and spectral decomposition.** For a Hilbert space H, a bounded linear operator $T \colon H \to H$ is *compact* if the image of its closed unit ball has compact closure. It is *self-adjoint* if

$$\langle Tf, \psi \rangle_H = \langle f, T\psi \rangle_H \qquad (f, \psi \in H).$$

A nonzero vector f is an *eigenvector* for the eigenvalue λ if $Tf = \lambda f$. For an operator on a function space we also call it an eigenfunction.

The *spectrum* $\operatorname{spec}(T)$ is the set of complex numbers λ for which $\lambda \operatorname{id} - T$ has no bounded inverse on the complexification of H. Here $\operatorname{id} \colon H_{\mathbb{C}} \to H_{\mathbb{C}}$ is the identity on that complexification, where $H_{\mathbb{C}} = H \oplus \sqrt{-1} H$. We use the complex-linear extension of T to $H_{\mathbb{C}}$, also denoted by T and defined by

$$T(f + \sqrt{-1}\psi) = Tf + \sqrt{-1} T\psi \quad (f, \psi \in H).$$

The same definition of spectrum applies to a bounded operator on any complex Banach space, with the identity of that space. For a self-adjoint operator this set is real. In finite dimension it is exactly the set of eigenvalues. In infinite dimension zero can belong to the spectrum without being an eigenvalue.

Theorem 7.2 (Riesz–Schauder and the self-adjoint spectral theorem, [30, Theorems 14.19 and 15.22])**.** *Let T be a compact self-adjoint operator on a separable real Hilbert space H. Then every nonzero spectral value is a real eigenvalue with finite-dimensional eigenspace. For each $a > 0$, there are only finitely many eigenvalues, counted with multiplicity, with absolute value* *greater than a. There is a finite or countable orthonormal family* $\{\psi_k\}_{k\in J}$ *indexed by a subset* $J \subset \mathbb{Z}_{\geq 0}$*, with nonzero eigenvalues* $\{\lambda_k\}_{k\in J}$ *such that*

$$H = \ker T \;\oplus\; \overline{\operatorname{span}}\{\psi_k \mid k \in J\}.$$

For every $f \in H$*,*

$$f = f_0 + \sum_{k\in J}\langle f, \psi_k\rangle\psi_k, \qquad Tf = \sum_{k\in J}\lambda_k\langle f, \psi_k\rangle\psi_k, \qquad f_0 \in \ker T,$$

where both sums converge in H*. If* J *is infinite, its eigenvalues tend to zero in any enumeration.*

This is the self-adjoint case of the spectral theorem for compact normal operators [30, Theorems 14.19 and 15.22]. In particular, a nonzero compact self-adjoint operator has a nonzero eigenvalue.

7.3. **Spectral perturbation and generalized eigenspaces.** Let E be a complex Banach space. A family of bounded linear operators $\{A_j : E \to E\}_{j\in\mathbb{Z}_{\geq 0}}$ is *collectively compact* if

$$\overline{\bigcup_{j\in\mathbb{Z}_{\geq 0}} A_j(B)} \text{ is compact}, \qquad B = \{v \in E \mid \|v\| \leq 1\}.$$

Strong convergence to a bounded linear operator $A : E \to E$ means that $\|A_j v - Av\| \to 0$ for every $v \in E$.

Theorem 7.3 (Riesz–Schauder, [30, Theorem 14.19])**.** *Let* $T : E \to E$ *be a compact linear operator on a complex Banach space. Then every* $\lambda \in \operatorname{spec}(T) \setminus \{0\}$ *is an eigenvalue, and*

$$0 < \dim\ker(T - \lambda\,\mathrm{id}) < \infty.$$

The nonzero spectrum is at most countable and has no accumulation point other than possibly zero.

Theorem 7.4 (Spectral inclusion, [1, Theorem 5.3])**.** *Let* $A_j, A : E \to E$ *be bounded linear operators on a complex Banach space. Assume that* $A_j \to A$ *strongly and that* $\{A_j - A\}_{j\in\mathbb{Z}_{\geq 0}}$ *is collectively compact. Then for every open set* $U \subset \mathbb{C}$ *containing* $\operatorname{spec}(A)$*, there is* $j_0 \in \mathbb{Z}_{\geq 0}$ *such that*

$$\operatorname{spec}(A_j) \subset U \quad (j \geq j_0).$$

Theorem 7.5 (Convergence of spectral projections, [1, Proposition 6.3])**.** *Under the operator hypotheses of Theorem 7.4, let* Γ *be a positively oriented finite union of rectifiable Jordan curves avoiding* $\operatorname{spec}(A)$ *and having winding number either zero or one at every point of that spectrum. The selected part consists precisely of the spectral points with winding number one. Define the Riesz projection* $\Pi : E \to E$ *by*

$$\Pi = \frac{1}{2\pi\sqrt{-1}}\int_\Gamma (\zeta\,\mathrm{id} - A)^{-1}\,d\zeta.$$

Then for all sufficiently large j, the same contour avoids $\operatorname{spec}(A_j)$*, and the projections* $\Pi_j \colon E \to E$ *defined by*

$$\Pi_j = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma (\zeta \operatorname{id} - A_j)^{-1}\, d\zeta$$

satisfy

$$\dim \operatorname{range} \Pi_j = \dim \operatorname{range} \Pi, \qquad \|\Pi_j v - \Pi v\| \to 0 \quad (v \in E).$$

If A_j and A are compact and Γ has winding number zero at 0, then $\dim \operatorname{range} \Pi_j$ *and* $\dim \operatorname{range} \Pi$ *are finite.*

For a bounded linear operator $T \colon E \to E$ on a complex Banach space and $\lambda \in \mathbb{C}$, we write

$$\mathcal{G}_\lambda(T) = \bigcup_{m \geq 1} \ker\big((T - \lambda \operatorname{id})^m\big)$$

for the generalized eigenspace at λ. If T is compact and $\lambda \in \operatorname{spec}(T) \setminus \{0\}$, then by [1, Section 7, p. 429], the Riesz projection $\Pi_{T,\lambda}$ associated with λ satisfies

$$\operatorname{range} \Pi_{T,\lambda} = \mathcal{G}_\lambda(T).$$

7.4. **Square roots of positive definite matrices.** We use the following result to construct orthonormal bases from Gram matrices.

Theorem 7.6 ([7, Theorems 7.29, 7.36 and 7.39])**.** *Every real symmetric matrix has an orthonormal basis of eigenvectors. If it is positive definite, then it has a unique symmetric positive definite square root. In an orthonormal eigenbasis, this square root is the diagonal matrix whose entries are the positive square roots of the original eigenvalues.*

8. Finite-dimensional approximation of distances

We now approximate the distance functions on a compact metric space by functions in a finite-dimensional vector space. The selected probability measure makes this vector space invariant under isometries. Using the spectral results from Section 7, we first prove approximation at one space. We then compare the resulting functions and coordinates on nearby spaces.

A simple construction explains why finite-dimensional coordinates can approximate a metric. Let (X, d) be nonempty and compact, fix $\varepsilon > 0$, and choose a finite ε-net $\{z_0, \dots, z_{m-1}\}$. Define

$$\Theta \colon X \to \mathbb{R}^m, \qquad \Theta(x) = \big(d(x, z_0), \dots, d(x, z_{m-1})\big).$$

For all $x, y \in X$,

$$d(x, y) - 2\varepsilon \leq \|\Theta(x) - \Theta(y)\|_\infty \leq d(x, y).$$

The upper bound follows from the triangle inequality. For the lower bound, choose i with $d(x, z_i) < \varepsilon$. Then

$$d(y, z_i) - d(x, z_i) \geq d(x, y) - 2d(x, z_i) > d(x, y) - 2\varepsilon.$$

Finite families of functions also underlie Shioya's approximation of metric measure spaces [33, author's manuscript, Section 4.4, Definition 4.37, Theorem 4.43 and Corollary 4.44]. More precisely, for every mm-space (X, d_X, μ_X) and every $\varepsilon > 0$, there are an integer $N \geq 1$ and real 1-Lipschitz functions $f_1, \dots, f_N$ on X such that, with $\Phi = (f_1, \dots, f_N)$,

$$\square\big(X, (\mathbb{R}^N, \|\cdot\|_\infty, \Phi_*\mu_X)\big) < \varepsilon.$$

Here $\square$ is the distance in Definition 2.18. The elementary estimate above concerns uniform error over all pairs of points of one compact space. Our construction also needs continuity throughout a Gromov–Hausdorff neighborhood and compatibility with isometries. A finite net at one space does not specify those choices. We obtain them from finite spectral subspaces, using the coefficients of unique best approximations as coordinates. For each compact metric space (Y, e), we define a norm N_Y on the coordinate space. Changing the orthonormal basis multiplies the coordinate vectors by an orthogonal matrix g and replaces the norm by $N_Y \circ g^{-1}$. Thus the distances between coordinate vectors do not depend on the basis.

Throughout this section, $\mathfrak{m}_{X,d}$ denotes the assignment of Theorem 5.6. The two subscripts record the carrier X and its metric d.

8.1. **The distance operator and its spectral subspaces.** Fix a nonempty compact metric space (X, d). The space

$$H_X = L^2(X, \mathfrak{m}_{X,d}), \qquad \langle f, \psi \rangle_{H_X} = \int_X f\psi \, d\mathfrak{m}_{X,d}$$

is a separable real Hilbert space. Here separability follows from density of continuous functions in L^2 and separability of $C(X)$. We define the distance operator $T_{X,d} \colon H_X \to H_X$ by

$$T_{X,d} f(x) = \int_X d(x,z) f(z) \, d\mathfrak{m}_{X,d}(z) \qquad (x \in X).$$

The integral is independent of the representative of f. This is the distance kernel operator of [28, Definition 11].

Lemma 8.1. *The operator $T_{X,d} \colon H_X \to H_X$ is compact and self-adjoint. It maps H_X into $C(X)$, and for every $f \in H_X$ and $x, y \in X$,*

$$\|T_{X,d} f\|_\infty \leq \operatorname{diam}(X,d) \|f\|_2, \qquad |T_{X,d} f(x) - T_{X,d} f(y)| \leq d(x,y) \|f\|_2. \tag{8.1}$$

Every eigenfunction with nonzero eigenvalue has a unique continuous representative.

Proof. By the Cauchy–Schwarz inequality [10, Lemma 3.2.27] and $\mathfrak{m}_{X,d}(X) = 1$, we obtain

$$\int_X |f| \, d\mathfrak{m}_{X,d} \leq \|f\|_2.$$

Consequently,

$$\begin{aligned}|T_{X,d}f(x)| &\le \int_X d(x,z)|f(z)|\,d\mathfrak{m}_{X,d}(z) \le \operatorname{diam}(X,d)\|f\|_2,\\ |T_{X,d}f(x)-T_{X,d}f(y)| &\le \int_X |d(x,z)-d(y,z)|\,|f(z)|\,d\mathfrak{m}_{X,d}(z)\\ &\le d(x,y)\|f\|_2.\end{aligned}$$

These estimates prove continuity of the image functions $T_{X,d}f$. They also show that the image of the unit ball of H_X is uniformly bounded and equicontinuous in $C(X)$. The Arzelà–Ascoli theorem (Theorem 2.8) and the continuous inclusion $C(X)\to H_X$ prove compactness. Symmetry of the kernel and Fubini's theorem [10, Theorem 2.8.20(a)] yield

$$\begin{aligned}\langle T_{X,d}f,\psi\rangle_{H_X} &= \int_X\int_X d(x,z)f(z)\psi(x)\,d\mathfrak{m}_{X,d}(z)\,d\mathfrak{m}_{X,d}(x)\\ &= \langle f, T_{X,d}\psi\rangle_{H_X}.\end{aligned}$$

These properties are also shown in [28, Propositions 12 and 13].

If $T_{X,d}f=\lambda f$ in H_X and $\lambda\ne 0$, then $\lambda^{-1}T_{X,d}f$ is a continuous representative of f. If two continuous functions agree almost everywhere, their difference cannot be nonzero at a point. Indeed, continuity would make its absolute value bounded below on a nonempty open set, and full support would make that set have positive measure. This proves uniqueness. $\square$

We always use these continuous representatives. Apply Theorem 7.2 to $T_{X,d}$ on H_X, and choose an orthonormal family $\{\psi_k\}_{k\in J}$ with corresponding nonzero eigenvalues $\{\lambda_k\}_{k\in J}$ as in that theorem. For $a>0$, put

$$J_{X,a}=\{k\in J \mid |\lambda_k|>a\},\qquad V_{X,a}=\operatorname{span}\{\psi_k \mid k\in J_{X,a}\}.$$

This is a finite-dimensional space of continuous functions. Define the orthogonal projection $P_{X,a}\colon H_X\to V_{X,a}$ by

$$P_{X,a}f=\sum_{k\in J_{X,a}}\langle f,\psi_k\rangle_{H_X}\psi_k. \tag{8.2}$$

Equivalently,

$$V_{X,a}=\bigoplus_{\substack{\lambda\in\operatorname{spec}(T_{X,d})\\|\lambda|>a}}\ker(T_{X,d}-\lambda\,\mathrm{id}),\qquad \operatorname{range}P_{X,a}=V_{X,a}.$$

These definitions do not depend on the orthonormal basis $\{\psi_k\}_{k\in J}$. Both positive and negative eigenvalues are retained. For every $f\in H_X$, the vector $P_{X,a}f$ converges in the norm of H_X to the orthogonal projection of f onto $(\ker T_{X,d})^{\perp}$ as $a\downarrow 0$.

8.2. **Approximation of distance functions.** For $x \in X$, we call the function

$$d_x = d(x, \cdot) \in C(X)$$

the *distance profile* of x. It records the distances from x to all points of X. The map

$$X \to C(X), \qquad x \longmapsto d_x$$

is an isometric embedding for the uniform norm, because

$$\|d_x - d_y\|_\infty = d(x, y).$$

By the triangle inequality, we have $|d_x(z) - d_y(z)| \le d(x, y)$ for every $z \in X$. Taking $z = x$, we obtain

$$\|d_x - d_y\|_\infty \ge |d_x(x) - d_y(x)| = d(x, y).$$

This is the distance-function construction of the Kuratowski embedding. Here the functions d_x are bounded because (X, d) is compact. The following lemma approximates all distance profiles in one finite spectral subspace.

Lemma 8.2. *For every nonempty compact metric space* (X, d) *and every* $\eta > 0$*, there is* $a > 0$ *such that* $a, -a \notin \operatorname{spec}(T_{X,d})$ *and*

$$\sup_{x \in X} \inf_{v \in V_{X,a}} \|d_x - v\|_\infty < \eta. \tag{8.3}$$

If X *has more than one point, then we may also require* $\dim V_{X,a} \ge 1$*.*

Proof. Let V be the closed linear span in $C(X)$ of the continuous representatives of all nonzero eigenfunctions. For $f \in H_X$, write $f = f_0 + f_\perp$ with $f_0 \in \ker T_{X,d}$ and $f_\perp \perp \ker T_{X,d}$. By the Riesz–Schauder and self-adjoint spectral theorem (Theorem 7.2),

$$\|P_{X,a} f - f_\perp\|_2 \to 0 \qquad (a \downarrow 0).$$

The continuous function $T_{X,d} f_0$ is zero almost everywhere and hence everywhere. Using (8.1),

$$\|T_{X,d} P_{X,a} f - T_{X,d} f\|_\infty \le \operatorname{diam}(X, d) \|P_{X,a} f - f_\perp\|_2 \to 0.$$

For every $a > 0$, we have

$$T_{X,d} P_{X,a} f \in V_{X,a} \subset V.$$

Since V is closed in the uniform norm, it follows that $T_{X,d} f \in V$.

Fix $x \in X$ and $r > 0$. Define a function $b_{x,r} \colon X \to [0, \infty)$ by

$$b_{x,r}(z) = \frac{\max\{1 - 2d(x, z)/r, 0\}}{\displaystyle\int_X \max\{1 - 2d(x, w)/r, 0\}\, d\mathfrak{m}_{X,d}(w)}.$$

The denominator is positive by full support. This is a continuous nonnegative function, its integral is one, and it vanishes outside $B(x, r; d)$. For every

$y \in X$,

$$|T_{X,d}b_{x,r}(y) - d_x(y)| = \left| \int_X (d(y,z) - d(y,x))b_{x,r}(z)\, d\mathfrak{m}_{X,d}(z) \right|$$
$$\leq \int_X d(x,z)b_{x,r}(z)\, d\mathfrak{m}_{X,d}(z) \leq r.$$

Thus d_x belongs to V.

Recall that

$$\|d_x - d_y\|_\infty = d(x,y),$$

as established before the lemma. Choose finitely many points $x_0, \ldots, x_{m-1}$ whose $\eta/3$-balls cover X. For each i, choose a finite sum v_i of nonzero eigenfunctions with $\|d_{x_i} - v_i\|_\infty < \eta/3$. Let Λ be the finite set of eigenvalues occurring in these sums. Choose $a > 0$ such that

$$a, -a \notin \operatorname{spec}(T_{X,d}), \qquad a < |\lambda| \quad (\lambda \in \Lambda).$$

Note that even if $\Lambda = \emptyset$, we can choose such a cutoff by requiring only $a, -a \notin \operatorname{spec}(T_{X,d})$. Such a cutoff exists because the nonzero spectrum is countable. Then every v_i belongs to $V_{X,a}$. For $d(x, x_i) < \eta/3$,

$$\inf_{v \in V_{X,a}} \|d_x - v\|_\infty \leq \|d_x - d_{x_i}\|_\infty + \|d_{x_i} - v_i\|_\infty < 2\eta/3.$$

This proves (8.3).

If X is not a singleton, choose $x, z \in X$ with $d(x,z) > 0$. The integral $T_{X,d}1(x)$ is positive because the distance from x is positive on a neighborhood of z of positive measure. Thus $T_{X,d} \neq 0$. By the Riesz–Schauder and self-adjoint spectral theorem (Theorem 7.2), it has a nonzero eigenvalue. If necessary, choose a smaller positive a below the absolute value of this eigenvalue and still satisfying $\pm a \notin \operatorname{spec}(T_{X,d})$. This retains the preceding approximation and ensures $\dim V_{X,a} \geq 1$. □

The uniform distance between the distance profiles is the original distance, but a best approximation in a finite-dimensional subspace need not be unique for the uniform norm. We shall minimize a finite L^p-norm instead. For $1 < p < \infty$, the best approximation exists and is unique by Theorem 7.1. Lemma 8.4 shows that the L^p-distances between distance profiles approximate the original distances uniformly when p is sufficiently large. We then combine this estimate with finite spectral approximation in (8.5).

To compare the best approximations, we use the following continuity statement for their coefficients.

Lemma 8.3. *Let $n \geq 1$, and let continuous functions $\mathcal{J}_j, \mathcal{J} \colon \mathbb{R}^n \to \mathbb{R}$ converge uniformly on every compact subset. Assume that each c_j minimizes $\mathcal{J}_j$, that these vectors form a bounded sequence, and that $\mathcal{J}$ has a unique minimizer c. Then $c_j \to c$.*

Proof. Every subsequence has a further convergent subsequence by boundedness. Write v for the limit of such a subsequence, which we relabel. Choose a closed Euclidean ball containing the relabeled subsequence and its limit v. On that ball,

$$|\mathcal{J}_j(c_j) - \mathcal{J}(v)| \le |\mathcal{J}_j(c_j) - \mathcal{J}(c_j)| \\ + |\mathcal{J}(c_j) - \mathcal{J}(v)| \longrightarrow 0.$$

The first term tends to zero by uniform convergence on the ball, and the second by continuity of the limiting function $\mathcal{J}$. For a fixed competitor $w \in \mathbb{R}^n$, minimality at each index and convergence at that competitor now imply

$$\mathcal{J}(v) = \lim \mathcal{J}_j(c_j) \le \lim \mathcal{J}_j(w) = \mathcal{J}(w).$$

Thus $v = c$ by uniqueness. For the sake of contradiction, suppose that the sequence $\{c_j\}_{j\in\mathbb{Z}_{\ge 0}}$ does not converge to c. Then there are $\varepsilon > 0$ and a subsequence satisfying $\|c_j - c\|_2 \ge \varepsilon$. A further subsequence converges by boundedness. Its limit must be c by the argument above, contradicting the displayed lower bound. Thus $c_j \to c$. ☐

Lemma 8.4. *Let (X, d) be a nonempty compact metric space and set $D = \operatorname{diam}(X, d)$. Then for every $r > 0$, the number*

$$c_r = \inf_{x\in X} \mathfrak{m}_{X,d}(B(x, r; d))$$

is positive. Moreover, for every $2 \le p < \infty$ and $x, y \in X$,

$$0 \le d(x, y) - \|d_x - d_y\|_{L^p(\mathfrak{m}_{X,d})} \le D(1 - c_r^{1/p}) + 2r. \tag{8.4}$$

Proof. Choose a finite $r/4$-net $\{z_0, \dots, z_{m-1}\}$. For each $x \in X$, some i satisfies $d(x, z_i) < r/4$, and then $B(z_i, r/4; d) \subset B(x, r; d)$. Therefore,

$$c_r \ge \min_{0\le i<m} \mathfrak{m}_{X,d}(B(z_i, r/4; d)) > 0.$$

For all $z \in X$, $|d_x(z) - d_y(z)| \le d(x, y)$. For $z \in B(x, r; d)$,

$$d_y(z) - d_x(z) \ge d(x, y) - 2d(x, z) > d(x, y) - 2r.$$

Consequently,

$$c_r^{1/p} \max\{d(x, y) - 2r, 0\} \le \|d_x - d_y\|_{L^p(\mathfrak{m}_{X,d})} \le d(x, y).$$

If $d(x, y) \le 2r$, then the error is at most $2r$, which is bounded by the right-hand side of (8.4). Otherwise the lower bound yields

$$\begin{aligned} d(x, y) &- \|d_x - d_y\|_p \\ &\le (1 - c_r^{1/p})d(x, y) + 2rc_r^{1/p} \\ &\le D(1 - c_r^{1/p}) + 2r. \end{aligned}$$

Here $0 < c_r \le 1$ because the measure $\mathfrak{m}_{X,d}$ is a probability. This proves the estimate in both cases. ☐

Fix $V_{X,a}$ and $2 \le p < \infty$. For each $x \in X$, let v_x minimize $v \mapsto \|d_x - v\|_p^p$ over $V_{X,a}$. The spectral subspace $V_{X,a}$ is finite-dimensional, and hence closed and convex in $L^p(X, \mathfrak{m}_{X,d})$. Theorem 7.1 therefore provides a unique minimizer. Here the L^p-norm is a genuine norm on the continuous functions in the spectral subspace $V_{X,a}$, by full support of $\mathfrak{m}_{X,d}$.

We call this function $v_x \in V_{X,a}$ the *finite-dimensional approximation of the distance profile*. The map $X \to V_{X,a}$, $x \mapsto v_x$, need not be injective. Its use here is to approximate distances with a controlled error.

If (8.3) holds, then $\|d_x - v_x\|_p < \eta$. Define $q_X\colon X \times X \to [0,\infty)$ by $q_X(x,y) = \|v_x - v_y\|_p$. The norm triangle inequality implies

$$|q_X(x,y) - \|d_x - d_y\|_p| \le \|v_x - d_x\|_p + \|v_y - d_y\|_p < 2\eta.$$

Together with Lemma 8.4, this yields

$$\|q_X - d\|_\infty \le 2\eta + D(1 - c_r^{1/p}) + 2r. \tag{8.5}$$

For any prescribed error, first choose r small, then choose a sufficiently large finite p, and finally choose η small and a corresponding spectral subspace.

8.3. **Continuity of finite spectral subspaces.** The operators for different spaces act on different Hilbert spaces. We compare them by extending the distance kernels to one compact ambient space. This produces operators on the fixed Banach space $C(Z, \mathbb{C})$, where we can apply the spectral inclusion theorem and the convergence theorem for spectral projections of Anselone and Palmer. These are Theorems 7.4 and 7.5 in Subsection 7.3. They preserve the cutoff gaps and the dimensions of the retained spectral subspaces, respectively.

By Lemma 2.17, the parameter-dependent integrals used below converge uniformly.

The next lemma compares the generalized eigenspaces of the ambient kernel operator with those of the self-adjoint distance operator.

Lemma 8.5. *Let Y be a nonempty compact subset of a compact metric space (Z, h), and put*

$$e = h|_{Y\times Y}, \qquad D = \operatorname{diam}(Z, h).$$

Regard $T_{Y,e}$ as its complex-linear extension to $L^2(Y, \mathfrak{m}_{Y,e}; \mathbb{C})$. Define

$$\mathsf{R}_Y\colon C(Z,\mathbb{C}) \to L^2(Y, \mathfrak{m}_{Y,e}; \mathbb{C}), \qquad \mathsf{R}_Y f = f|_Y,$$

$$\mathsf{S}_Y\colon L^2(Y, \mathfrak{m}_{Y,e}; \mathbb{C}) \to C(Z,\mathbb{C}), \quad \mathsf{S}_Y\psi(z) = \int_Y h(z,w)\psi(w)\, d\mathfrak{m}_{Y,e}(w).$$

Set $B_Y = \mathsf{S}_Y\mathsf{R}_Y$. Then B_Y is compact. For every $\lambda \in \mathbb{C}\setminus\{0\}$ and every integer $m \ge 1$, we have

$$\mathcal{G}_\lambda(B_Y) = \ker\big((B_Y - \lambda\,\mathrm{id})^m\big) = \ker(B_Y - \lambda\,\mathrm{id}),$$
$$\mathcal{G}_\lambda(T_{Y,e}) = \ker\big((T_{Y,e} - \lambda\,\mathrm{id})^m\big) = \ker(T_{Y,e} - \lambda\,\mathrm{id}).$$

Restriction induces the linear isomorphism

$$\mathsf{R}_Y : \mathcal{G}_\lambda(B_Y) \to \mathcal{G}_\lambda(T_{Y,e}),$$

whose inverse is the restriction of $\lambda^{-1}\mathsf{S}_Y$. *Moreover,*

$$\operatorname{spec}(B_Y) \setminus \{0\} = \operatorname{spec}(T_{Y,e}) \setminus \{0\} \subset [-D, D].$$

Proof. Since $\mathfrak{m}_{Y,e}$ is a probability measure,

$$\|\mathsf{R}_Y f\|_2 \le \|f\|_\infty.$$

The Cauchy–Schwarz inequality also yields

$$\|\mathsf{S}_Y \psi\|_\infty \le D\|\psi\|_2,$$
$$|\mathsf{S}_Y\psi(z) - \mathsf{S}_Y\psi(z_0)| \le h(z, z_0)\|\psi\|_2.$$

Thus R_Y and S_Y are bounded, and the Arzelà–Ascoli theorem (Theorem 2.8) implies that S_Y is compact. Consequently, B_Y is compact. By their definitions,

$$B_Y = \mathsf{S}_Y\mathsf{R}_Y, \qquad T_{Y,e} = \mathsf{R}_Y\mathsf{S}_Y, \qquad \mathsf{R}_Y B_Y = T_{Y,e}\mathsf{R}_Y. \tag{8.6}$$

The intertwining identity is displayed in Figure 2.

$$\begin{tikzcd}
C(Z,\mathbb{C}) \arrow[r, "B_Y"] \arrow[d, "\mathsf{R}_Y"'] & C(Z,\mathbb{C}) \arrow[d, "\mathsf{R}_Y"] \\
L^2(Y,\mathfrak{m}_{Y,e};\mathbb{C}) \arrow[r, "T_{Y,e}"'] \arrow[ur, "\mathsf{S}_Y"] & L^2(Y,\mathfrak{m}_{Y,e};\mathbb{C})
\end{tikzcd}$$

FIGURE 2. Restriction intertwines the ambient operator B_Y and the distance operator $T_{Y,e}$. The factorizations $B_Y = \mathsf{S}_Y\mathsf{R}_Y$ and $T_{Y,e} = \mathsf{R}_Y\mathsf{S}_Y$ explain the comparison of their nonzero spectral subspaces.

Fix $\lambda \in \mathbb{C}\setminus\{0\}$ and an integer $m \ge 1$. Restriction is injective on $\ker\big((B_Y - \lambda\,\mathrm{id})^m\big)$. Indeed, if $\mathsf{R}_Y f = 0$, then $B_Y f = 0$, and

$$0 = (B_Y - \lambda\,\mathrm{id})^m f = (-\lambda)^m f$$

forces $f = 0$. The intertwining identity and the self-adjoint spectral theorem (Theorem 7.2) imply

$$\begin{aligned}\mathsf{R}_Y \ker\big((B_Y - \lambda\,\mathrm{id})^m\big) &\subset \ker\big((T_{Y,e} - \lambda\,\mathrm{id})^m\big)\\ &= \ker(T_{Y,e} - \lambda\,\mathrm{id}).\end{aligned}$$

Indeed, in the orthogonal spectral decomposition of $T_{Y,e}$, raising a spectral value minus λ to a positive power changes neither its vanishing nor its nonvanishing. Conversely, if $T_{Y,e}\psi = \lambda\psi$, then

$$f = \lambda^{-1}\mathsf{S}_Y\psi, \qquad \mathsf{R}_Y f = \psi, \qquad B_Y f = \lambda f.$$

To prove that the ambient generalized eigenspace contains no additional vectors, take $f \in \ker\big((B_Y - \lambda\,\mathrm{id})^m\big)$ and put

$$\psi = \mathsf{R}_Y f, \qquad v = \lambda^{-1}\mathsf{S}_Y\psi.$$

The preceding identities imply

$$(B_Y - \lambda\,\mathrm{id})v = 0, \qquad \mathsf{R}_Y(f - v) = 0.$$

Both f and v belong to $\ker\big((B_Y - \lambda\,\mathrm{id})^m\big)$. Injectivity of restriction on this kernel therefore implies $f = v$. Thus

$$\ker\big((B_Y - \lambda\,\mathrm{id})^m\big) = \ker(B_Y - \lambda\,\mathrm{id}).$$

In particular, restriction induces the linear isomorphism

$$\mathsf{R}_Y\colon \ker(B_Y - \lambda\,\mathrm{id}) \to \ker(T_{Y,e} - \lambda\,\mathrm{id}),$$

whose inverse is the restriction of $\lambda^{-1}\mathsf{S}_Y$. Both operators are compact, so the Riesz–Schauder theorem (Theorem 7.3) ensures that every nonzero spectral value is an eigenvalue. By Lemma 8.1, $\|T_{Y,e}\| \le \mathrm{diam}(Y, e) \le D$. Thus

$$\mathrm{spec}(B_Y) \setminus \{0\} = \mathrm{spec}(T_{Y,e}) \setminus \{0\} \subset [-D, D].$$

□

The following lemma constructs spectral projections on the fixed space $C(Z,\mathbb{C})$ and compares them with the orthogonal projections on the varying Hilbert spaces.

Lemma 8.6. *Let X_j, X be nonempty compact subsets of a compact metric space (Z, h), with restricted metrics*

$$d_j = h|_{X_j \times X_j}, \qquad d = h|_{X\times X},$$

such that

$$\mathsf{H}[h](X_j, X) \to 0, \qquad \mathfrak{m}_{X_j,d_j} \to \mathfrak{m}_{X,d} \quad \textit{weakly on } Z.$$

Fix $a > 0$ with $a, -a \notin \mathrm{spec}(T_{X,d})$. Use the operators of Lemma 8.5 and set

$$A_j = B_{X_j}, \qquad A = B_X.$$

Then for all sufficiently large j,

$$\pm a \notin \mathrm{spec}(T_{X_j,d_j}), \qquad \dim V_{X_j,a} = \dim V_{X,a} = n.$$

Moreover, there are finite-rank Riesz projections

$$\Pi_j, \Pi\colon C(Z,\mathbb{C}) \to C(Z,\mathbb{C})$$

associated with the eigenvalues of absolute value greater than a of A_j and A, respectively, which preserve real functions and satisfy

$$\|\Pi_j f - \Pi f\|_\infty \to 0 \quad (f \in C(Z,\mathbb{C})).$$

Put $\Pi_X = \Pi$ and $\Pi_{X_j} = \Pi_j$. For $Y = X$ or $Y = X_j$ with j sufficiently large, put $e = h|_{Y\times Y}$. Then restriction induces the complex-linear isomorphism

$$\mathsf{R}_Y\colon \mathrm{range}\,\Pi_Y \to V_{Y,a} + \sqrt{-1}V_{Y,a},$$

and

$$(\Pi_Y f)|_Y = P_{Y,a}(f|_Y) \quad (f \in C(Z,\mathbb{C})).$$

Here $P_{Y,a}$ *also denotes the complex-linear extension of the orthogonal projection onto* $V_{Y,a}$.

Proof. By definition,

$$A_j f(z) = \int_Z h(z,w) f(w)\, d\mathfrak{m}_{X_j,d_j}(w),$$
$$Af(z) = \int_Z h(z,w) f(w)\, d\mathfrak{m}_{X,d}(w).$$

For each fixed f, the family $\{h(z,\cdot)f(\cdot) \mid z \in Z\}$ is compact in $C(Z,\mathbb{C})$. By Lemma 2.17 with parameter $z \in Z$,

$$\|A_j f - Af\|_\infty \to 0.$$

Thus $A_j \to A$ strongly. Writing $D = \operatorname{diam}(Z,h)$, we also have, for $\|f\|_\infty \le 1$,

$$\|A_j f\|_\infty \le D, \qquad |A_j f(z) - A_j f(z_0)| \le h(z,z_0).$$

Hence $\bigcup_j A_j\{f \mid \|f\|_\infty \le 1\}$ has compact closure by the Arzelà–Ascoli theorem (Theorem 2.8). Thus $\{A_j\}_{j\in\mathbb{Z}_{\ge 0}}$ is collectively compact. The same estimates with bounds $2D$ and 2 apply to $A_j - A$, so $\{A_j - A\}_{j\in\mathbb{Z}_{\ge 0}}$ is collectively compact as well.

By Lemma 8.5 and the hypothesis on a, we have

$$\operatorname{spec}(A) \subset \mathbb{C} \setminus \{-a, a\}.$$

The set on the right is open. The preceding estimates verify the strong convergence and collective compactness required in Theorem 7.4. Applying that theorem with this open set, we obtain $\pm a \notin \operatorname{spec}(A_j)$ for all sufficiently large j. Lemma 8.5 then implies

$$\pm a \notin \operatorname{spec}(T_{X_j,d_j}).$$

This proves the exclusion of the cutoff values.

We next choose one contour that encloses the eigenvalues of absolute value greater than a for both A and A_j. Choose $L > \max\{D,a\} + 1$ and let Ω be the union of the two rectangles

$$\{\zeta \in \mathbb{C} \mid a < \operatorname{Re}\zeta < L,\ |\operatorname{Im}\zeta| < 1\},$$
$$\{\zeta \in \mathbb{C} \mid -L < \operatorname{Re}\zeta < -a,\ |\operatorname{Im}\zeta| < 1\}.$$

Write $\Gamma = \partial\Omega$ with positive orientation. By Lemma 8.5, all the spectral values of A and A_j are real and lie in $[-D,D]$. Thus a spectral value can meet Γ only at a or $-a$, and we have already excluded these two values. Consequently, the same contour Γ avoids both spectra for all sufficiently large j and encloses exactly their spectral values of absolute value greater than a. We also have $0 \notin \overline{\Omega}$. Figure 3 shows the two components of this contour.

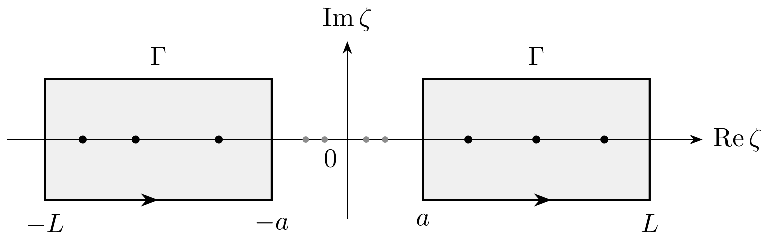

FIGURE 3. The positively oriented boundary $\Gamma = \partial\Omega$ encloses the retained positive and negative spectral values. The dots are schematic. The gaps at $\pm a$ allow the same contour to be used for A_j for all sufficiently large j.

Define the Riesz projections $\Pi, \Pi_j \colon C(Z,\mathbb{C}) \to C(Z,\mathbb{C})$ by

$$\Pi = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma (\zeta \,\mathrm{id} - A)^{-1}\, d\zeta, \qquad \Pi_j = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma (\zeta \,\mathrm{id} - A_j)^{-1}\, d\zeta.$$

For $f \in C(Z,\mathbb{C})$ and each fixed $z \in Z$, we have

$$(\Pi f)(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \big[(\zeta \,\mathrm{id} - A)^{-1} f\big](z)\, d\zeta.$$

The integration variable ζ ranges over the complex contour Γ, while the point $z \in Z$ is fixed. Evaluation at z is a bounded linear functional on $C(Z,\mathbb{C})$, so it commutes with the contour integral. The same formula holds for Π_j with A_j in place of A. The Riesz projections satisfy

$$\begin{aligned}\operatorname{range} \Pi &= \bigoplus_{\lambda \in \operatorname{spec}(A) \cap \Omega} \mathcal{G}_\lambda(A) \\ &= \bigoplus_{\lambda \in \operatorname{spec}(A) \cap \Omega} \ker(A - \lambda \,\mathrm{id}),\end{aligned}$$

and, for all sufficiently large j,

$$\begin{aligned}\operatorname{range} \Pi_j &= \bigoplus_{\lambda \in \operatorname{spec}(A_j) \cap \Omega} \mathcal{G}_\lambda(A_j) \\ &= \bigoplus_{\lambda \in \operatorname{spec}(A_j) \cap \Omega} \ker(A_j - \lambda \,\mathrm{id}).\end{aligned}$$

In each display, the second equality follows from Lemma 8.5. The sums are finite by Theorem 7.3, since $0 \notin \overline{\Omega}$. We can now apply Theorem 7.5 on the fixed Banach space $C(Z,\mathbb{C})$. The operator hypotheses were verified at the beginning of the proof, and the preceding construction verifies the hypotheses on Γ. We obtain

$$\dim \operatorname{range} \Pi_j = \dim \operatorname{range} \Pi, \qquad \|\Pi_j f - \Pi f\|_\infty \to 0 \quad (f \in C(Z,\mathbb{C})) \tag{8.7}$$

for all sufficiently large j. The dimensions are finite because $0 \notin \overline{\Omega}$. The convergence in (8.7) means uniform convergence on Z for each fixed function $f \in C(Z, \mathbb{C})$.

We now compare the projections on $C(Z, \mathbb{C})$ with the spectral subspaces of the distance operators. For $Y = X$ or $Y = X_j$ with j sufficiently large, write $\Pi_X = \Pi$ and $\Pi_{X_j} = \Pi_j$. Since Ω selects exactly the eigenvalues of absolute value greater than a, Lemma 8.5 induces the complex-linear isomorphism

$$\mathsf{R}_Y \colon \operatorname{range} \Pi_Y \to V_{Y,a} + \sqrt{-1} V_{Y,a}.$$

Here the space on the right is the complexification of the real space $V_{Y,a}$. Its complex dimension equals $\dim_{\mathbb{R}} V_{Y,a}$. The dimension equality in (8.7) therefore yields

$$\begin{aligned} \dim_{\mathbb{R}} V_{X_j,a} &= \dim_{\mathbb{C}} \operatorname{range} \Pi_j = \dim_{\mathbb{C}} \operatorname{range} \Pi \\ &= \dim_{\mathbb{R}} V_{X,a} = n. \end{aligned}$$

This also applies when $n = 0$.

We next prove the identity relating projection and restriction for an arbitrary continuous function. The identity (8.6) implies

$$(\zeta \operatorname{id} - T_{Y,e}) \mathsf{R}_Y = \mathsf{R}_Y (\zeta \operatorname{id} - B_Y).$$

For $\zeta \in \Gamma$, both operators in parentheses are invertible. Multiplying by their inverses on the left and on the right, respectively, we obtain

$$\mathsf{R}_Y (\zeta \operatorname{id} - B_Y)^{-1} = (\zeta \operatorname{id} - T_{Y,e})^{-1} \mathsf{R}_Y \quad (\zeta \in \Gamma).$$

Both sides are bounded maps from $C(Z, \mathbb{C})$ to $L^2(Y, \mathfrak{m}_{Y,e}; \mathbb{C})$. Both resolvents in this identity depend continuously on $\zeta \in \Gamma$ in the operator norm. The contour integrals can therefore be computed as limits of Riemann sums. Since R_Y is bounded and linear, it commutes with these sums and their limits. For every $f \in C(Z, \mathbb{C})$, we consequently obtain

$$\begin{aligned} \mathsf{R}_Y \Pi_Y f &= \mathsf{R}_Y \left(\frac{1}{2\pi\sqrt{-1}} \int_\Gamma (\zeta \operatorname{id} - B_Y)^{-1} f \, d\zeta \right) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \mathsf{R}_Y (\zeta \operatorname{id} - B_Y)^{-1} f \, d\zeta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma (\zeta \operatorname{id} - T_{Y,e})^{-1} \mathsf{R}_Y f \, d\zeta. \end{aligned}$$

The first equality is the definition of Π_Y, the second uses the bounded linearity of R_Y, and the third uses the preceding resolvent identity.

We next identify the last integral with $P_{Y,a}(\mathsf{R}_Y f)$. Here $P_{Y,a}$ also denotes the complex-linear extension of the orthogonal projection defined in (8.2). Let ψ be an eigenvector of $T_{Y,e}$ with eigenvalue λ. Then

$$(\zeta \operatorname{id} - T_{Y,e})^{-1} \psi = \frac{1}{\zeta - \lambda} \psi \quad (\zeta \in \Gamma).$$

Since Γ encloses exactly the eigenvalues with absolute value greater than a, we have

$$\begin{aligned}&\frac{1}{2\pi\sqrt{-1}}\int_\Gamma (\zeta \operatorname{id} - T_{Y,e})^{-1}\psi \, d\zeta \\ &\quad = \left(\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \frac{d\zeta}{\zeta - \lambda}\right)\psi = \begin{cases}\psi, & |\lambda| > a,\\ 0, & |\lambda| < a.\end{cases}\end{aligned}$$

The scalar integral equals one when λ is inside Γ and zero when it is outside. The second case includes $\lambda = 0$, so the contour integral vanishes on $\ker T_{Y,e}$. By the self-adjoint spectral theorem (Theorem 7.2),

$$\begin{aligned}&\overline{\ker T_{Y,e} \oplus \bigoplus_{\substack{\lambda \in \operatorname{spec}(T_{Y,e})\\ \lambda \neq 0}} \ker(T_{Y,e} - \lambda \operatorname{id})} \\ &\quad = L^2(Y, \mathfrak{m}_{Y,e}; \mathbb{C}),\end{aligned}$$

where the closure is taken in the L^2-norm. The preceding calculation shows that the contour integral and $P_{Y,a}$ agree on each summand. Both operators are bounded, so they agree on the whole Hilbert space. Substituting this into the preceding calculation and using $\mathsf{R}_Y f = f|_Y$, we conclude that

$$(\Pi_Y f)|_Y = P_{Y,a}(f|_Y). \tag{8.8}$$

On the left we project in $C(Z,\mathbb{C})$ and then restrict to Y. On the right we first restrict and then apply the projection on $L^2(Y, \mathfrak{m}_{Y,e}; \mathbb{C})$. Both sides of (8.8) are continuous on Y. Since $\mathfrak{m}_{Y,e}$ has full support, their equality in L^2 also implies pointwise equality on Y.

Finally, the distance kernels are real and Ω is invariant under complex conjugation. Conjugating the contour integrals and reversing their orientation therefore yields

$$\overline{\Pi_Y f} = \Pi_Y \overline{f}.$$

Thus Π and Π_j preserve real functions. $\square$

Lemma 8.7. *Let X_j, X be compact subsets of a compact metric space (Z, h), with restricted metrics*

$$d_j = h|_{X_j \times X_j}, \qquad d = h|_{X \times X},$$

such that

$$\mathsf{H}[h](X_j, X) \to 0, \qquad \mathfrak{m}_{X_j,d_j} \to \mathfrak{m}_{X,d} \quad \textit{weakly on } Z.$$

Fix $a > 0$ with $a, -a \notin \operatorname{spec}(T_{X,d})$. Then for all sufficiently large j,

$$a, -a \notin \operatorname{spec}(T_{X_j,d_j}), \qquad \dim V_{X_j,a} = \dim V_{X,a} = n.$$

If $n > 0$, then for every real orthonormal basis $\{\psi_k\}_{0\le k<n}$ of $V_{X,a}$, there are real orthonormal bases $\{\psi_{j,k}\}_{0\le k<n}$ of $V_{X_j,a}$ for all sufficiently large j such that

$$\max_{0\le k<n} \sup_{(x_j,x)\in R_j} |\psi_{j,k}(x_j) - \psi_k(x)| \to 0 \tag{8.9}$$

for every sequence of correspondences $R_j \subset X_j \times X$ *satisfying*

$$\sup_{(x_j,x)\in R_j} h(x_j,x) \to 0.$$

Moreover, there are continuous functions

$$\widehat{\psi}_{j,k}, \widehat{\psi}_k \colon Z \to \mathbb{R} \quad (0 \le k < n)$$

satisfying

$$\widehat{\psi}_{j,k}|_{X_j} = \psi_{j,k}, \qquad \widehat{\psi}_k|_X = \psi_k, \qquad \max_{0\le k<n} \|\widehat{\psi}_{j,k} - \widehat{\psi}_k\|_\infty \to 0.$$

Proof. Lemma 8.6 proves the exclusion of $\pm a$ from the spectra and the dimension equality. Let Π_j, Π be the projections constructed in that lemma. There is nothing more to prove when $n = 0$. Assume that $n > 0$. To construct the real orthonormal bases $\{\psi_{j,k}\}_{0\le k<n}$ of $V_{X_j,a}$ satisfying (8.9), we first project continuous extensions of the prescribed basis $\{\psi_k\}_{0\le k<n}$. The resulting functions need not be orthonormal for the varying measures $\mathfrak{m}_{X_j,d_j}$. A Gram matrix corrects this without changing their limit. By the Tietze–Urysohn theorem (Theorem 2.19), choose continuous real extensions $\widetilde{\psi}_k \colon Z \to \mathbb{R}$ of the prescribed functions $\psi_k \colon X \to \mathbb{R}$, and put

$$u_{j,k} = \Pi_j \widetilde{\psi}_k, \qquad u_k = \Pi \widetilde{\psi}_k.$$

Applying (8.7) to each fixed function $\widetilde{\psi}_k$, we obtain $\|u_{j,k} - u_k\|_\infty \to 0$ for every $0 \le k < n$. Moreover, (8.8) implies

$$\mathsf{R}_X u_k = P_{X,a} \mathsf{R}_X \widetilde{\psi}_k = P_{X,a} \psi_k = \psi_k.$$

Thus $u_k|_X = \psi_k$. The same identity on X_j shows that

$$u_{j,k}|_{X_j} = P_{X_j,a}(\widetilde{\psi}_k|_{X_j}) \in V_{X_j,a}.$$

Define the real Gram matrix

$$G_j = \left(\int_Z u_{j,k} u_{j,l} \, d\mathfrak{m}_{X_j,d_j} \right)_{0\le k,l<n}.$$

The products $u_{j,k}u_{j,l}$ converge uniformly on Z to $u_k u_l$. Since $u_k|_X = \psi_k$ and the prescribed basis is orthonormal,

$$\int_Z u_k u_l \, d\mathfrak{m}_{X,d} = \delta_{kl}.$$

Each measure $\mathfrak{m}_{X_j,d_j}$ has total mass one. Hence

$$\begin{aligned} &|(G_j)_{kl} - \delta_{kl}| \\ &\quad \le \|u_{j,k}u_{j,l} - u_k u_l\|_\infty \\ &\qquad + \left| \int_Z u_k u_l \, d\mathfrak{m}_{X_j,d_j} - \int_Z u_k u_l \, d\mathfrak{m}_{X,d} \right|. \end{aligned}$$

The first term tends to zero by uniform convergence, and the second tends to zero by weak convergence. It follows that

$$(G_j)_{kl} \to \int_X \psi_k \psi_l \, d\mathfrak{m}_{X,d} = \delta_{kl}.$$

Thus $G_j \to I_n$, and for large j the matrix G_j is positive definite. We spell out the finite-dimensional fact used to normalize it. Theorem 7.6 defines $G_j^{-1/2}$ by replacing each positive eigenvalue with its inverse square root in an orthonormal eigenbasis. Write

$$\varepsilon_j = \|G_j - I_n\|_{\mathrm{op}}, \qquad \|M\|_{\mathrm{op}} = \sup_{\|c\|_2=1} \|Mc\|_2$$

for the Euclidean operator norm. Entrywise convergence in this fixed dimension implies $\varepsilon_j \to 0$. Choose $j_0 \in \mathbb{Z}_{\geq 0}$ such that

$$0 \leq \varepsilon_j < 1 \quad (j \geq j_0).$$

Since G_j is real symmetric, we have

$$\mathrm{spec}(G_j) \subset [1-\varepsilon_j, 1+\varepsilon_j] \subset (0,\infty) \quad (j \geq j_0).$$

For these indices, Theorem 7.6 implies

$$\begin{aligned}
\|G_j^{-1/2} - I_n\|_{\mathrm{op}} &= \max_{\lambda \in \mathrm{spec}(G_j)} |\lambda^{-1/2} - 1| \\
&\leq \max_{1-\varepsilon_j \leq t \leq 1+\varepsilon_j} |t^{-1/2} - 1| \\
&= \max_{1-\varepsilon_j \leq t \leq 1+\varepsilon_j} \frac{|t-1|}{\sqrt{t}(1+\sqrt{t})} \\
&\leq \frac{\varepsilon_j}{\sqrt{1-\varepsilon_j}(1+\sqrt{1-\varepsilon_j})} \longrightarrow 0.
\end{aligned}$$

Thus

$$G_j^{-1/2} \xrightarrow[j\to\infty]{\|\cdot\|_{\mathrm{op}}} I_n.$$

Set

$$\widetilde{\psi}_{j,k} = \sum_{l=0}^{n-1} u_{j,l} (G_j^{-1/2})_{lk}, \qquad \psi_{j,k} = \widetilde{\psi}_{j,k}|_{X_j}.$$

The Gram matrix of the family $\{\psi_{j,k}\}_{0 \leq k < n}$ is $G_j^{-1/2} G_j G_j^{-1/2} = I_n$. By (8.8), the functions $\psi_{j,k}$ belong to $V_{X_j,a}$. The displayed identity makes them orthonormal, and their number equals $\dim V_{X_j,a}$, so they form an orthonormal basis of $V_{X_j,a}$. Moreover, $\widetilde{\psi}_{j,k} \to u_k$ uniformly on Z. If $\delta_j = \sup_{R_j} h(x_j, x)$, then

$$\sup_{R_j} |\psi_{j,k}(x_j) - \psi_k(x)| \leq \|\widetilde{\psi}_{j,k} - u_k\|_\infty + \sup_{\substack{z,z_0 \in Z \\ h(z,z_0) \leq \delta_j}} |u_k(z) - u_k(z_0)| \to 0.$$

Taking $\widehat{\psi}_{j,k} = \widetilde{\psi}_{j,k}$ and $\widehat{\psi}_k = u_k$ proves the final assertion as well. $\square$

8.4. **Local maps to normed spaces.** We now turn the spectral subspace into coordinates in $\mathbb{R}^n$. A choice of orthonormal basis determines the coordinates, but the resulting pseudometric is independent of this choice.

We use the orthogonal group

$$O(n) = \{g \in \mathbb{R}^{n\times n} \mid g^{\mathsf{T}} g = I_n\},$$

acting on column vectors in $\mathbb{R}^n$. Thus g preserves the Euclidean norm and $g^{-1} = g^{\mathsf{T}}$.

To approximate the metrics on compact spaces in a sufficiently small Gromov–Hausdorff neighborhood, we construct a continuous map $\mathbf{v}_Y : Y \to \mathbb{R}^n$ and a norm N_Y for each compact metric space (Y, e) in that neighborhood. We choose the map $\mathbf{v}_Y$ and the norm N_Y so that

$$N_Y\big(\mathbf{v}_Y(y) - \mathbf{v}_Y(z)\big) \approx e(y, z) \quad (y, z \in Y),$$

with the uniform error specified below. Along each convergent sequence in this neighborhood, we can choose orthonormal coordinates so that the maps $\mathbf{v}_Y$ converge uniformly along correspondences whose paired points become uniformly close in a common compact metric space, and the norms N_Y converge uniformly on the Euclidean unit sphere. We shall use the linear structure of $\mathbb{R}^n$ to combine the approximations by a partition of unity in Section 11.

Theorem 8.8. *Let $(X, d) \in \mathcal{M}$ be a nonempty compact metric space and let $b > 0$. Then there are an open subset $\mathcal{W} \subset \mathcal{M}$ with $(X, d) \in \mathcal{W}$ and an integer $n \geq 1$ with the following properties. For each representative $(Y, e) \in \mathcal{W}$, there are a norm N_Y on $\mathbb{R}^n$ and a continuous map $\mathbf{v}_Y : Y \to \mathbb{R}^n$. The norm N_Y and the map $\mathbf{v}_Y$ are expressed in an orthonormal coordinate system. If this coordinate system is changed by an orthogonal matrix g, then the new coordinate map is $g \circ \mathbf{v}_Y$ and the new norm is $N_Y \circ g^{-1}$. In formulas,*

$$(\mathbf{v}_Y, N_Y) \longmapsto (g \circ \mathbf{v}_Y, N_Y \circ g^{-1}) \qquad (g \in O(n)). \tag{8.10}$$

For every isometry $u : (Y, e) \to (Y_0, e_0)$ between representatives in $\mathcal{W}$, there is $g \in O(n)$ such that

$$\mathbf{v}_{Y_0} \circ u = g \circ \mathbf{v}_Y, \qquad N_{Y_0} = N_Y \circ g^{-1}. \tag{8.11}$$

The norm N_Y and the map $\mathbf{v}_Y$ satisfy the following conditions.

(A1) *For every $y \in Y$,*

$$N_Y(\mathbf{v}_Y(y)) \leq 2 \operatorname{diam}(Y, e).$$

(A2) *Let $(Y_j, e_j), (Y, e) \in \mathcal{W}$ have isometric embeddings into a compact (Z, h) with $\mathsf{H}[h](Y_j, Y) \to 0$. Then the coordinates can be chosen so that*

$$\sup_{\|c\|_2 = 1} |N_{Y_j}(c) - N_Y(c)| \to 0, \qquad \sup_{(y_j, y) \in R_j} \|\mathbf{v}_{Y_j}(y_j) - \mathbf{v}_Y(y)\|_2 \to 0$$

for every sequence of correspondences $R_j \subset Y_j \times Y$ satisfying

$$\sup_{(y_j,y)\in R_j} h(y_j,y) \to 0.$$

(A3) *The formula*

$$q_Y(y,z) = N_Y(\mathbf{v}_Y(y) - \mathbf{v}_Y(z))$$

defines an isometry-invariant continuous pseudometric on Y. In particular, for every isometry as above,

$$q_{Y_0}(u(y),u(z)) = q_Y(y,z) \quad (y,z \in Y).$$

The function

$$\mathcal{E}\colon \mathcal{W} \to [0,\infty), \qquad \mathcal{E}(Y,e) = \sup_{y,z\in Y} |q_Y(y,z) - e(y,z)|$$

is continuous and satisfies $\mathcal{E}(X,d) < b$.

We call these data on $\mathcal{W}$ a *local model*. The map $\mathbf{v}_Y$ need not be injective, which is why q_Y is a pseudometric.

Proof. For the singleton center, assign to every $(Y,e) \in \mathcal{M}$ the data $n = 1$, $N_Y = |\cdot|$, and $\mathbf{v}_Y = 0$. Then $q_Y = 0$ and $\mathcal{E}(Y,e) = \operatorname{diam}(Y,e)$. The error is continuous because the diameter is continuous for $\mathcal{GH}$. For a prescribed $b > 0$, we may take its domain to be the open neighborhood

$$\mathcal{W} = \{(Y,e) \in \mathcal{M} \mid \operatorname{diam}(Y,e) < b\} \ni *.$$

On this domain the zero maps and fixed norm satisfy (A1)–(A3), with the error equal to the diameter. The coordinate maps $\mathbf{v}_Y$ vanish and the norm N_Y is fixed, so the coordinate and isometry identities also hold. This proves the theorem when the center is a singleton.

Assume henceforth that the center is not a singleton. We construct the local data from one spectral cutoff, prove their continuity first on each carrier and then as the carrier varies, and finally verify invariance and the error estimate.

Step 1. A neighborhood with constant spectral dimension. Choose $2 \le p < \infty$ and a spectral cutoff a at (X,d) so that (8.5) is less than b, with $a, -a \notin \operatorname{spec}(T_{X,d})$ and $\dim V_{X,a} = n \ge 1$. Define

$$\mathcal{W} = \{(Y,e) \in \mathcal{M} \mid a, -a \notin \operatorname{spec}(T_{Y,e}),\ \dim V_{Y,a} = n\}.$$

Then $(X,d) \in \mathcal{W}$. To prove that $\mathcal{W}$ is open, let $(Y_j,e_j) \to (Y,e) \in \mathcal{W}$ in $\mathcal{M}$. By Lemma 2.7, realize these spaces as compact subsets of a common compact metric space (Z,h) with

$$\mathsf{H}[h](Y_j,Y) \to 0.$$

Theorem 5.6 implies

$$\mathfrak{m}_{Y_j,e_j} \to \mathfrak{m}_{Y,e} \quad \text{weakly on } Z.$$

Lemma 8.6 now yields $(Y_j,e_j) \in \mathcal{W}$ for all sufficiently large j. Since $\mathcal{M}$ is metrizable, this proves openness.

Step 2. Norms and continuous coordinate maps. For $(Y, e) \in \mathcal{W}$, choose a real orthonormal basis $\psi_0, \dots, \psi_{n-1}$ of $V_{Y,a}$. Define the norm $N_Y : \mathbb{R}^n \to [0, \infty)$ by

$$N_Y(c) = \left(\int_Y \left| \sum_{k=0}^{n-1} c_k \psi_k(z) \right|^p d\mathfrak{m}_{Y,e}(z) \right)^{1/p}. \tag{8.12}$$

We define $\mathbf{v}_Y : Y \to \mathbb{R}^n$ by taking $\mathbf{v}_Y(y)$ to be the unique minimizer over $c \in \mathbb{R}^n$ of the function $\mathcal{J}_Y : Y \times \mathbb{R}^n \to [0, \infty)$ defined by

$$\mathcal{J}_Y(y, c) = \int_Y \left| e(y, z) - \sum_{k=0}^{n-1} c_k \psi_k(z) \right|^p d\mathfrak{m}_{Y,e}(z). \tag{8.13}$$

Existence and uniqueness of this minimizer follow from Theorem 7.1, applied to the finite-dimensional spectral subspace of

$$L^p(Y, \mathfrak{m}_{Y,e}).$$

Thus $\sum_k (\mathbf{v}_Y(y))_k \psi_k$ is the finite-dimensional approximation of the distance profile constructed above. The bold vector $\mathbf{v}_Y(y)$ records its coordinates in the chosen orthonormal basis $\{\psi_k\}_{0 \le k < n}$. Figure 4 displays this construction of $\mathbf{v}_Y$.

$$\begin{tikzcd}
e(y, \cdot) \in L^p(Y, \mathfrak{m}_{Y,e}) \arrow[d, "\text{best approximation in } L^p"] \\
\displaystyle\sum_{k=0}^{n-1} (\mathbf{v}_Y(y))_k \psi_k \in V_{Y,a} \arrow[d, "\text{coordinates in } \{\psi_k\}_{0 \le k < n}"] \\
\mathbf{v}_Y(y) \in (\mathbb{R}^n, N_Y)
\end{tikzcd}$$

FIGURE 4. The best approximation of $e(y, \cdot)$ in $V_{Y,a}$ has coordinate vector $\mathbf{v}_Y(y)$ in the chosen L^2-orthonormal basis. This vector is the minimizer in (8.13). With the L^p norm on $V_{Y,a}$, the coordinate map in the second arrow is a linear isometry onto $(\mathbb{R}^n, N_Y)$ by (8.12).

Since $p \ge 2$ and the measure $\mathfrak{m}_{Y,e}$ is a probability, orthonormality implies

$$N_Y(c) \ge \left\| \sum_{k=0}^{n-1} c_k \psi_k \right\|_2 = \|c\|_2.$$

Write $e_y = e(y,\cdot) \in C(Y)$ for the distance profile at y. By minimality,

$$\left\| e_y - \sum_{k=0}^{n-1} (\mathbf{v}_Y(y))_k \psi_k \right\|_p \le \|e_y\|_p \le \operatorname{diam}(Y,e).$$

The triangle inequality therefore proves

$$\|\mathbf{v}_Y(y)\|_2 \le N_Y(\mathbf{v}_Y(y)) \le 2\operatorname{diam}(Y,e). \tag{8.14}$$

This proves (A1). In particular, the minimizers stay in a fixed compact Euclidean ball when the diameters are bounded.

For $M > 0$, write

$$B_M = \{c \in \mathbb{R}^n \mid \|c\|_2 \le M\}.$$

For fixed Y, the function $\mathcal{J}_Y$ is continuous. If $y_l \to y$, continuity on the product of the compact space Y and each B_M implies

$$\sup_{c\in B_M} |\mathcal{J}_Y(y_l, c) - \mathcal{J}_Y(y,c)| \to 0.$$

The bound (8.14) and Lemma 8.3 prove continuity of $\mathbf{v}_Y$.

Step 3. Convergence of the objectives and norms. Let $Y_j \to Y$ in the stated compact ambient space. Write $e_j = h|_{Y_j\times Y_j}$ and $e = h|_{Y\times Y}$. By Theorem 5.6,

$$\mathfrak{m}_{Y_j,e_j} \to \mathfrak{m}_{Y,e} \quad \text{weakly on } Z.$$

Apply Lemma 8.7 to the basis $\{\psi_k\}_{0\le k<n}$ of $V_{Y,a}$ chosen in Step 2. We obtain real orthonormal bases $\{\psi_{j,k}\}_{0\le k<n}$ of $V_{Y_j,a}$ and continuous functions $\widetilde{\psi}_{j,k}, \widetilde{\psi}_k \colon Z \to \mathbb{R}$ such that

$$\widetilde{\psi}_{j,k}|_{Y_j} = \psi_{j,k}, \qquad \widetilde{\psi}_k|_Y = \psi_k, \qquad \max_{0\le k<n} \|\widetilde{\psi}_{j,k} - \widetilde{\psi}_k\|_\infty \to 0.$$

Define $\mathcal{J}_j, \mathcal{J} \colon Z\times\mathbb{R}^n \to [0,\infty)$ by

$$\mathcal{J}_j(z,c) = \int_Z \left| h(z,w) - \sum_{k=0}^{n-1} c_k \widetilde{\psi}_{j,k}(w)\right|^p d\mathfrak{m}_{Y_j,e_j}(w),$$
$$\mathcal{J}(z,c) = \int_Z \left| h(z,w) - \sum_{k=0}^{n-1} c_k \widetilde{\psi}_{k}(w)\right|^p d\mathfrak{m}_{Y,e}(w).$$

By (8.13),

$$\mathcal{J}_j|_{Y_j\times\mathbb{R}^n} = \mathcal{J}_{Y_j}, \qquad \mathcal{J}|_{Y\times\mathbb{R}^n} = \mathcal{J}_Y.$$

For every $M > 0$,

$$\sup_{\substack{z\in Z\\ \|c\|_2\le M}} |\mathcal{J}_j(z,c) - \mathcal{J}(z,c)| \to 0. \tag{8.15}$$

Apply Lemma 2.17 with compact parameter space $Z \times \{c \in \mathbb{R}^n \mid \|c\|_2 \le M\}$. Uniform convergence of the extended basis functions $\widetilde{\psi}_{j,k}$ to $\widetilde{\psi}_k$ implies uniform convergence of the displayed integrands on this product with Z. For the norms N_{Y_j}, N_Y, apply the same lemma with the Euclidean unit sphere $\mathbb{S}^{n-1}$ as parameter space and integrands $\left|\sum_k c_k \widetilde{\psi}_{j,k}(w)\right|^p$. It follows that

$$\sup_{\|c\|_2=1} |N_{Y_j}(c)^p - N_Y(c)^p| \to 0.$$

For nonnegative real numbers s, t and $p \geq 1$,

$$|s^{1/p} - t^{1/p}| \leq |s-t|^{1/p}.$$

Applying this inequality to the powers of N_{Y_j} and N_Y proves uniform convergence on the Euclidean unit sphere $\mathbb{S}^{n-1}$.

Step 4. Uniform convergence of the coordinate maps. We now prove the second convergence in (A2) for the bases $\{\psi_{j,k}\}_{0\leq k<n}$ and $\{\psi_k\}_{0\leq k<n}$ chosen in Step 3. For the sake of contradiction, suppose that this convergence fails. Then after passing to a subsequence, there are $\varepsilon > 0$ and pairs $(y_j, y_{0,j}) \in R_j$ with $\|\mathbf{v}_{Y_j}(y_j) - \mathbf{v}_Y(y_{0,j})\|_2 \geq \varepsilon$. Compactness of Y and (8.14) permit a further subsequence such that

$$y_{0,j} \to y \in Y, \qquad \mathbf{v}_{Y_j}(y_j) \to c \in \mathbb{R}^n.$$

The condition on the correspondences implies

$$h(y_j, y) \leq h(y_j, y_{0,j}) + h(y_{0,j}, y) \longrightarrow 0.$$

For every $M > 0$, (8.15) and continuity of $\mathcal{J}$ on $Z \times B_M$ imply

$$\begin{aligned} &\sup_{c\in B_M} |\mathcal{J}_j(y_j, c) - \mathcal{J}(y,c)| \\ &\quad \leq \sup_{\substack{z\in Z \\ c\in B_M}} |\mathcal{J}_j(z,c) - \mathcal{J}(z,c)| \\ &\qquad + \sup_{c\in B_M} |\mathcal{J}(y_j,c) - \mathcal{J}(y,c)| \longrightarrow 0. \end{aligned}$$

Lemma 8.3 and (8.14) imply $c = \mathbf{v}_Y(y)$. But continuity of $\mathbf{v}_Y$ also implies $\mathbf{v}_Y(y_{0,j}) \to \mathbf{v}_Y(y)$, contradicting the lower bound. This proves (A2).

Step 5. Changes of basis and isometries. If a second basis is written as $\widehat{\psi}_k = \sum_l g_{kl}\psi_l$ with $g \in O(n)$, then its coefficient vectors and norms satisfy

$$\widehat{\mathbf{v}}_Y(y) = g\mathbf{v}_Y(y), \qquad \widehat{N}_Y(c) = N_Y(g^{-1}c).$$

This proves (8.10). For an isometry $u\colon (Y,e) \to (Y_0, e_0)$, the pullback map

$$U\colon L^2(Y_0, \mathfrak{m}_{Y_0,e_0}) \to L^2(Y, \mathfrak{m}_{Y,e}), \qquad f \longmapsto f\circ u$$

is unitary because $u_*\mathfrak{m}_{Y,e} = \mathfrak{m}_{Y_0,e_0}$. It intertwines the distance operators,

$$T_{Y,e}(f\circ u) = (T_{Y_0,e_0}f)\circ u.$$

Thus

$$U\big(V_{Y_0,a}\big) = V_{Y,a}.$$

Let $\{\psi_{0,k}\}_{0\le k<n}$ be the orthonormal basis chosen on Y_0. Since both $\{\psi_{0,k}\circ u\}_k$ and $\{\psi_k\}_k$ are orthonormal bases of $V_{Y,a}$, there is $g\in O(n)$ such that

$$\psi_{0,k}\circ u=\sum_{l=0}^{n-1} g_{kl}\psi_l.$$

Using this identity and $u_*\mathfrak{m}_{Y,e}=\mathfrak{m}_{Y_0,e_0}$, we obtain

$$\begin{aligned} N_{Y_0}(c)&=N_Y(g^{-1}c),\\ \mathcal{J}_{Y_0}(u(y),c)&=\mathcal{J}_Y(y,g^{-1}c)\quad (y\in Y,\ c\in\mathbb{R}^n). \end{aligned}$$

The second identity and uniqueness of the minimizers imply

$$\mathbf{v}_{Y_0}(u(y))=g\mathbf{v}_Y(y).$$

This proves both identities in (8.11). Consequently,

$$\begin{aligned} q_{Y_0}(u(y),u(z))&=N_{Y_0}\big(g(\mathbf{v}_Y(y)-\mathbf{v}_Y(z))\big)\\ &=q_Y(y,z). \end{aligned}$$

Thus the pseudometric q_Y is independent of the coordinates and is invariant under isometries of (Y,e).

Step 6. Continuity of the approximation error. The formula in (A3) defines a continuous pseudometric because it pulls back the distance of a normed space by the continuous map $\mathbf{v}_Y$. Step 5 proves its invariance. Set

$$\begin{aligned} \eta_j&=\sup_{\|c\|_2=1}|N_{Y_j}(c)-N_Y(c)|,\\ \delta_j&=\sup_{(y_j,y)\in R_j}\|\mathbf{v}_{Y_j}(y_j)-\mathbf{v}_Y(y)\|_2. \end{aligned}$$

Steps 3 and 4 show that both numbers tend to zero. Choose $M>0$ which bounds all the coordinate norms in (8.14) along the sequence and on the limit space. Such a bound exists because the diameters converge. For two corresponding pairs, write

$$\begin{aligned} v_j&=\mathbf{v}_{Y_j}(y_j)-\mathbf{v}_{Y_j}(z_j),\\ v&=\mathbf{v}_Y(y)-\mathbf{v}_Y(z). \end{aligned}$$

Then $\|v_j\|_2\le 2M$ and $\|v_j-v\|_2\le 2\delta_j$. Homogeneity and the norm triangle inequality imply

$$\begin{aligned} &|N_{Y_j}(v_j)-N_Y(v)|\\ &\quad\le |N_{Y_j}(v_j)-N_Y(v_j)|+N_Y(v_j-v)\\ &\quad\le 2M\eta_j+2\delta_j\max_{\|c\|_2=1}N_Y(c). \end{aligned}$$

The right-hand side is independent of the corresponding pairs and tends to zero. Thus

$$\Delta_j:=\sup_{(y_j,y),(z_j,z)\in R_j}|q_{Y_j}(y_j,z_j)-q_Y(y,z)|\to 0.$$

If $\gamma_j = \sup_{R_j} h(y_j, y)$, then

$$|e_j(y_j, z_j) - e(y, z)| \le 2\gamma_j.$$

Applying the uniform-norm estimate in Lemma 2.9,

$$|\mathcal{E}(Y_j, e_j) - \mathcal{E}(Y, e)| \le \Delta_j + 2\gamma_j \to 0.$$

The bound at (X, d) was arranged in Step 1 by (8.5). This proves (A3). $\square$

8.5. **Homotopy density of doubling spaces.** The local models can be combined before taking metric quotients. This produces simultaneous approximations of all compact metric spaces, with continuous dependence on both the space and the approximation scale. In this subsection, we write Y for a compact metric space with distance d_Y. Let $\mathcal{F} \subset \mathcal{M}$ consist of the spaces admitting an isometric embedding into a finite-dimensional real normed space. The dimension and the norm may depend on the space.

A subset D of a space T is *homotopy dense* if there is a continuous map $H \colon T \times [0,1] \to T$ such that $H(\cdot, 0) = \mathrm{id}_T$ and $H(T \times (0,1]) \subset D$. We prove a quantitative form of this property for $\mathcal{F}$.

Corollary 8.9. *There is a continuous map $H \colon \mathcal{M} \times [0,1] \to \mathcal{M}$ such that*

$$H(Y,0) = Y, \qquad H(Y,t) \in \mathcal{F}, \qquad \mathcal{GH}(H(Y,t), Y) < t/2 \quad (0 < t \le 1).$$

Near each point of $\mathcal{M} \times (0,1]$, the dimensions of the normed spaces realizing the images have a common finite bound. Moreover, $H(Y,t)$ is connected whenever Y is connected, and path connected whenever Y is path connected.

Proof. We combine the local pseudometrics by a partition of unity, prove continuity of their metric quotients, and realize each quotient in a finite-dimensional normed space.

Step 1. Combining local pseudometrics. Put $P = \mathcal{M} \times (0,1]$. For each $(Y_0, t_0) \in P$, apply Theorem 8.8 with tolerance $t_0/4$, and let W be the open neighborhood supplied by that theorem. Continuity of the error function permits an open neighborhood $V \subset W \times (0,1]$ of (Y_0, t_0) on which the error is less than the time coordinate. Since P is metrizable, choose a locally finite partition of unity $\{\lambda_i\}_{i \in I}$ subordinate to such neighborhoods V_i, with $\operatorname{supp} \lambda_i \subset V_i$. Denote the corresponding local models by $(W_i, n_i, v_{i,Y}, N_{i,Y})$, and define $q_{i,Y} \colon Y \times Y \to [0,\infty)$ and $E_i \colon W_i \to [0,\infty)$ by

$$q_{i,Y}(y,z) = N_{i,Y}(v_{i,Y}(y) - v_{i,Y}(z)), \qquad E_i(Y) = \|q_{i,Y} - d_Y\|_\infty.$$

Thus $E_i(Y) < t$ whenever $(Y,t) \in V_i$. For $(Y,t) \in P$, define $q_{Y,t} \colon Y \times Y \to [0,\infty)$ and $H \colon P \to \mathcal{M}$ by

$$q_{Y,t}(y,z) = \sum_{i \in I} \lambda_i(Y,t) q_{i,Y}(y,z), \qquad H(Y,t) = [Y, q_{Y,t}].$$

A term with zero weight is defined to be zero, even outside the domain of its local model. The sum is finite at each parameter and is a continuous pseudometric on Y. The isometry invariance in Theorem 8.8 makes its metric quotient independent of the chosen representative of Y. Furthermore,

$$\|q_{Y,t} - d_Y\|_\infty \le \sum_{i\,:\,\lambda_i(Y,t)>0} \lambda_i(Y,t)E_i(Y) < t.$$

Lemma 2.9 proves the asserted GH estimate.

Step 2. Continuity of the quotients. Let $(Y_j, t_j) \to (Y, t)$ in P. By local finiteness, there are an open neighborhood $U \subset P$ of (Y,t) and a finite set $I_0 \subset I$ such that

$$\lambda_i|_U = 0 \qquad (i \in I \setminus I_0).$$

For all sufficiently large j, we have $(Y_j, t_j) \in U$, so only indices in I_0 occur in the sums below. By Lemma 2.7, realize the spaces in a common compact metric space and choose correspondences $R_j \subset Y_j \times Y$ whose paired points have ambient distances tending uniformly to zero. For an index with $\lambda_i(Y,t) > 0$, condition (A2) of Theorem 8.8 permits coordinates in which the point maps v_{i,Y_j} converge uniformly to $v_{i,Y}$ along R_j and the norms N_{i,Y_j} converge uniformly to $N_{i,Y}$ on the Euclidean unit sphere $\mathbb{S}^{n_i-1}$. To see explicitly that the pseudometrics converge, write

$$\eta_j = \sup_{\|u\|_2=1} |N_{i,Y_j}(u) - N_{i,Y}(u)|, \qquad M_i = \sup_{\|u\|_2=1} N_{i,Y}(u).$$

Since $N_{i,Y}$ is a norm on the finite-dimensional space $\mathbb{R}^{n_i}$, we have

$$m_i = \min_{\|u\|_2=1} N_{i,Y}(u) > 0.$$

For all sufficiently large j, we have $\eta_j \le m_i/2$, and homogeneity implies

$$N_{i,Y_j}(u) \ge (m_i - \eta_j)\|u\|_2 \ge \frac{m_i}{2}\|u\|_2 \quad (u \in \mathbb{R}^{n_i}).$$

By condition (A1),

$$\sup_{y_j \in Y_j} \|v_{i,Y_j}(y_j)\|_2 \le \frac{4 \operatorname{diam} Y_j}{m_i}.$$

For corresponding difference vectors $w_j = v_{i,Y_j}(y_j) - v_{i,Y_j}(z_j)$ and $w = v_{i,Y}(y) - v_{i,Y}(z)$,

$$|N_{i,Y_j}(w_j) - N_{i,Y}(w)| \le \eta_j\|w_j\|_2 + M_i\|w_j - w\|_2.$$

In particular,

$$\sup_{y_j, z_j \in Y_j} \|w_j\|_2 \le \frac{8 \operatorname{diam} Y_j}{m_i},$$

and $\operatorname{diam} Y_j \to \operatorname{diam} Y$. The expression $\eta_j\|w_j\|_2 + M_i\|w_j - w\|_2$ therefore tends uniformly to zero over pairs in R_j. For an index with $\lambda_i(Y,t) = 0$, condition (A1) instead implies

$$0 \le \lambda_i(Y_j, t_j) q_{i,Y_j}(y_j, z_j) \le 4\lambda_i(Y_j, t_j) \operatorname{diam} Y_j \longrightarrow 0$$

whenever the weight is positive. This also covers parameters crossing a model domain boundary. Since diameters remain bounded, all weighted terms converge uniformly along the correspondences. Thus

$$\sup_{(y_j,y),(z_j,z)\in R_j} |q_{Y_j,t_j}(y_j,z_j) - q_{Y,t}(y,z)| \longrightarrow 0.$$

Lemma 2.9 proves continuity on P. Set $H(Y,0)=Y$. The estimate

$$\mathcal{GH}(H(Y_j,t_j),Y) \le t_j/2 + \mathcal{GH}(Y_j,Y)$$

proves continuity at time zero as well.

Step 3. Finite-dimensional realization. Fix $(Y,t)\in P$ and let $J = \{i \mid \lambda_i(Y,t)>0\}$. This is a finite nonempty set. On $\bigoplus_{i\in J}\mathbb{R}^{n_i}$, the formula

$$N((a_i)_{i\in J}) = \sum_{i\in J} \lambda_i(Y,t) N_{i,Y}(a_i)$$

defines a norm. All its weights are strictly positive. The compact image of the continuous map

$$Y \to \bigoplus_{i\in J}\mathbb{R}^{n_i}, \qquad y \longmapsto (v_{i,Y}(y))_{i\in J}$$

is isometric to $H(Y,t)$. Its ambient dimension is $\sum_{i\in J} n_i$. Choose U and I_0 near (Y,t) as in Step 2. For every $(Z,s)\in U$, we have

$$\{i\in I \mid \lambda_i(Z,s)>0\}\subset I_0, \qquad \sum_{i\,:\,\lambda_i(Z,s)>0} n_i \le \sum_{i\in I_0} n_i.$$

Thus the integer $\sum_{i\in I_0} n_i$ bounds the ambient dimensions throughout U. Finally, this image is a continuous image of Y, so connectedness and path connectedness are preserved. □

Corollary 8.10. *Let K be a compact metrizable space, $f\colon K\to\mathcal{M}$ continuous, and $\varepsilon>0$. Then there are an integer $N\ge 1$ and a continuous map $g\colon K\to\mathcal{M}$ such that*

$$\sup_{k\in K}\mathcal{GH}(f(k),g(k)) < \varepsilon,$$

and each $g(k)$ admits an isometric embedding into a normed space of dimension at most N. The norms may vary with k.

Proof. Choose $0<\tau\le 1$ with $\tau/2<\varepsilon$ and set $g(k)=H(f(k),\tau)$. The locally bounded dimension in Corollary 8.9 has a finite bound on the compact set $f(K)\times\{\tau\}$. The same corollary proves continuity and the uniform error bound. □

For a finite subset $S\subset X$ of a metric space (X,d) with $|S|\ge 2$, write

$$\operatorname{sep}(S) = \min\{d(x,y) \mid x,y\in S,\ x\ne y\}.$$

We use the finite-set description of the *Assouad dimension* $\dim_{\mathrm{A}} X$. It is the infimum of the exponents $\beta \geq 0$ for which some $C \geq 1$ satisfies

$$|S| \leq C \left(\frac{\operatorname{diam} S}{\operatorname{sep}(S)} \right)^{\beta} \tag{8.16}$$

for every such finite subset of X. The infimum of the empty set is infinite. A metric space is *doubling* if its balls can be covered by a uniformly bounded number of balls of half the radius. This is equivalent to finite Assouad dimension, as recalled in the author's [20, Section 2.2 and Lemma 2.3].

Lemma 8.11. *If a metric space X admits an isometric embedding into an n-dimensional real normed space, where $n \geq 1$, then every finite subset $S \subset X$ with at least two points satisfies*

$$|S| \leq \left(1 + \frac{2 \operatorname{diam} S}{\operatorname{sep}(S)} \right)^{n} \leq 3^{n} \left(\frac{\operatorname{diam} S}{\operatorname{sep}(S)} \right)^{n}.$$

In particular, $\dim_{\mathrm{A}} X \leq n$.

Proof. Identify S with its isometric image in $(\mathbb{R}^n, N)$, and write

$$B_N(x, r) = \{ v \in \mathbb{R}^n \mid N(v - x) < r \}.$$

Put $a = \operatorname{sep}(S)$ and $D = \operatorname{diam} S$. Choose $x_0 \in S$. The open balls $\{B_N(x, a/2)\}_{x \in S}$ are pairwise disjoint. They are contained in the ball of radius $D + a/2$ centered at x_0. Let vol denote Lebesgue volume on $\mathbb{R}^n$. Translation invariance and scaling imply

$$\begin{aligned} |S|(a/2)^n \operatorname{vol}(B_N(0,1)) &= \sum_{x \in S} \operatorname{vol}(B_N(x, a/2)) \\ &\leq \operatorname{vol}(B_N(x_0, D + a/2)) \\ &= (D + a/2)^n \operatorname{vol}(B_N(0,1)). \end{aligned}$$

Since $0 < \operatorname{vol}(B_N(0,1)) < \infty$, division proves the first bound. The second follows from $D/a \geq 1$. The definition of Assouad dimension now applies with $C = 3^n$ and $\beta = n$. $\square$

Corollary 8.12. *The doubling compact metric spaces form a homotopy dense subset of $\mathcal{M}$. The same assertion holds relative to the subspace of connected compact metric spaces and relative to the subspace of path connected compact metric spaces.*

Proof. By Lemma 8.11, every member of $\mathcal{F}$ is doubling. Apply Corollary 8.9 and its connectedness assertions. $\square$

The author proved that the set of all doubling compact metric spaces is a dense meager F_σ subset of $\mathcal{M}$ in [20, Theorem 1.1]. Corollary 8.12 adds simultaneous continuous approximation of families to this description. The relative statements apply to the connected and path connected classes studied in [21].

The approximations may also be required to be uniformly perfect. We use the convention that a bounded space X is *uniformly perfect* if some $0 < c < 1$ has the following property. For every $x \in X$ and $0 < r < \operatorname{diam} X$, there is $y \in X$ with $cr < d(x,y) \le r$. A space is *uniformly disconnected* if some $\delta > 0$ satisfies

$$\max_{0 \le k < m} d(x_k, x_{k+1}) \ge \delta d(x_0, x_m)$$

for every finite chain $x_0, \dots, x_m$ with $m \ge 1$.

Corollary 8.13. *The compact metric spaces that are doubling, uniformly perfect, and not uniformly disconnected form a homotopy dense subset of* $\mathcal{M}$. *This also holds relative to the connected and path connected subspaces.*

Proof. For $t > 0$, put $J(Y,t) = H(Y,t) \times [0, t/2]$ with the maximum metric, and set $J(Y,0) = Y$. The product of a correspondence R between A and B with the correspondence $\{(as, bs) \mid 0 \le s \le 1\}$ between the intervals has distortion at most $\max\{\operatorname{dis}(R), |a-b|\}$. Consequently,

$$\mathcal{GH}(A \times [0,a], B \times [0,b]) \le \max\{\mathcal{GH}(A,B), |a-b|/2\}.$$

This proves continuity at positive times. Collapsing the interval factor also shows that

$$\mathcal{GH}(J(Y,t), Y) < t/2 + t/4 = 3t/4,$$

which proves continuity at time zero. Adding one real coordinate with the maximum norm realizes $J(Y,t)$ in a finite-dimensional normed space, so it is doubling.

Write $A = H(Y,t)$, $a = t/2$, and $D = \max\{\operatorname{diam} A, a\}$. Let d denote the maximum metric on $A \times [0,a]$. For $(x,u) \in A \times [0,a]$ and $0 < r < D$, a point on the same interval fiber can be chosen at distance $\min\{r, a/2\}$. Indeed, one endpoint of the interval is at distance at least $a/2$ from u. With $c = a/(4D) \in (0,1)$, we have

$$cr < \min\{r, a/2\} \le r \qquad (0 < r < D).$$

This proves uniform perfectness. Let $\delta > 0$ and choose an integer $m > 1/\delta$. For a fixed $x \in A$, the chain $z_k = (x, ka/m)$, where $0 \le k \le m$, satisfies

$$d(z_0, z_m) = a, \qquad \max_{0 \le k < m} d(z_k, z_{k+1}) = a/m < \delta a.$$

This violates the chain inequality for every fixed $\delta > 0$, so the product is not uniformly disconnected. Connectedness and path connectedness are preserved by the interval product and by H. □

8.6. **Dependence of the model dimension on accuracy.** The common finite dimension in Corollary 8.10 may depend on the error tolerance. The packing estimate above explains why this dependence cannot be removed, even for a single compact space homeomorphic to a convergent sequence.

Proposition 8.14. *Let $N \geq 1$ be an integer and let $A_j \to X$ in the Gromov–Hausdorff topology. If every A_j admits an isometric embedding into a normed space of dimension at most N, then $\dim_{\mathrm{A}} X \leq N$. Consequently, if $\dim_{\mathrm{A}} X = \infty$ and A_j admits an isometric embedding into an n_j-dimensional normed space, then $n_j \to \infty$.*

Proof. Fix a finite subset $S = \{x_0, \dots, x_{m-1}\} \subset X$ with $m \geq 2$, and put $D = \operatorname{diam} S$ and $a = \operatorname{sep}(S) > 0$. Choose correspondences $R_j \subset A_j \times X$ with $\operatorname{dis}(R_j) \to 0$. For each $0 \leq k < m$, choose $x_{j,k} \in A_j$ with $(x_{j,k}, x_k) \in R_j$. For sufficiently large j, these points are distinct because

$$d_{A_j}(x_{j,k}, x_{j,l}) \geq a - \operatorname{dis}(R_j) > 0 \quad (0 \leq k, l < m,\ k \neq l).$$

Their sets $S_j = \{x_{j,0}, \dots, x_{j,m-1}\}$ satisfy

$$|\operatorname{diam} S_j - D| \leq \operatorname{dis}(R_j), \qquad |\operatorname{sep}(S_j) - a| \leq \operatorname{dis}(R_j).$$

Lemma 8.11 therefore implies

$$m \leq 3^N \left(\frac{\operatorname{diam} S_j}{\operatorname{sep}(S_j)} \right)^N \longrightarrow 3^N (D/a)^N.$$

Thus (8.16) holds on X with $C = 3^N$ and $\beta = N$. This proves the first assertion. For the sake of contradiction, suppose that n_j does not tend to infinity. Then there are a subsequence and an integer $N \geq 1$ such that $n_j \leq N$ along that subsequence. The first assertion applied to that subsequence contradicts $\dim_{\mathrm{A}} X = \infty$. $\square$

The uniformity of the packing constant with respect to the ambient norm is essential here. Finite Assouad dimension of the individual approximating spaces alone does not impose a finite Assouad dimension on their GH limit. The author's work on Assouad dimension and convergence contains a convergent-sequence space with infinite Assouad dimension [19, Theorem 1.4 and Remark 4.1]. For the present application, the following direct instance of the telescope construction in [19, Section 4.1] suffices.

Example 8.15. For every integer $m \geq 2$, let E_m be a set of m points, with these sets pairwise disjoint, and put $r_m = 2^{-m}$. Define

$$X = \{\infty\} \sqcup \bigsqcup_{m \geq 2} E_m.$$

The distances between distinct points are specified by

$$d(x, y) = \begin{cases} r_m, & x, y \in E_m, \\ \max\{r_m, r_k\}, & x \in E_m,\ y \in E_k,\ m \neq k, \\ r_m, & x \in E_m,\ y = \infty, \end{cases}$$

with symmetry and zero diagonal understood. This is an ultrametric. Indeed, for a triangle meeting two different sets, the two largest distances coincide, and triangles within one set have equal nonzero distances. The same verification applies to triangles containing ∞. Every point other than

∞ is isolated, and every neighborhood of ∞ contains all but finitely many points. Hence X is compact and homeomorphic to a convergent sequence together with its limit. It has topological dimension zero, since the isolated singletons and the tails containing ∞ form a clopen basis. It also has Hausdorff dimension zero because it is countable. Nevertheless,

$$|E_m| = m, \qquad \frac{\operatorname{diam} E_m}{\operatorname{sep}(E_m)} = 1.$$

No fixed constants in (8.16) can satisfy these inequalities for all m. Thus $\dim_{\mathrm{A}} X = \infty$. By Proposition 8.14, the ambient dimensions of finite-dimensional normed approximations to X must tend to infinity as their GH errors tend to zero.

Part III. The topology of Gromov–Hausdorff space III: The absolute retract property

Abstract. We prove that the Gromov–Hausdorff space of isometry classes of nonempty compact metric spaces is an absolute retract for all metrizable spaces. Starting from continuous finite-dimensional local approximations, we represent the varying norms in fixed Banach spaces and combine the local maps by a partition of unity. Hyperspaces modulo compact groups provide absolute retracts through which the identity map can be approximated with arbitrarily small homotopy tracks. Hanner's domination theorem implies the absolute neighborhood retract property. Contractibility then implies that every continuous map from a closed subset of an arbitrary metrizable space into the Gromov–Hausdorff space extends to the whole domain.

Keywords. Gromov–Hausdorff space, absolute retract, absolute extensor, hyperspace, approximation.

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9. Introduction to Part III

The Gromov–Hausdorff space contains compact metric spaces with different underlying topologies and arbitrarily large dimensions. Let $\mathcal{M}$ denote the space of their isometry classes with the ordinary unpointed Gromov–Hausdorff distance $\mathcal{GH}$. We prove that $\mathcal{M}$ is an absolute retract for metrizable spaces. Antonyan records the question whether $\mathcal{M}$ is an absolute retract in [5, p. 2]. For each positive integer n, he identifies the subspace represented by compact subsets of $\mathbb{R}^n$ with the Hilbert cube minus a point [6, Corollary 5.3].

The relation between Gromov–Hausdorff geometry and hyperspaces is useful for this question. Mémoli and Wan [29, arXiv v2, Theorem 1] prove that every Gromov–Hausdorff geodesic can be realized as a Hausdorff geodesic of compact subsets of one compact metric space. Curtis [12, Theorem 1.6] proves, in particular, that the hyperspace of nonempty compact subsets of a Banach space, equipped with the Hausdorff metric of its norm, is an absolute retract.

In the present paper, we approximate the identity map of $\mathcal{M}$ through absolute retracts obtained from hyperspaces modulo compact groups. The approximation homotopies have arbitrarily small point tracks. This construction proves the following theorem.

Theorem 9.1. *The space $\mathcal{M}$ is an absolute retract for all metrizable spaces.*

We prove the equivalent absolute extensor assertion in Theorem 11.4. For every metrizable space S, every closed subset $A \subset S$, and every continuous map $f\colon A \to \mathcal{M}$, there is a continuous map $F\colon S \to \mathcal{M}$ with $F|_A = f$. No dimension or separability restriction is imposed on S.

The absolute retract argument uses the local finite-dimensional maps and varying norms of Theorem 8.8, the main result of Part II, including their continuity and compatibility with isometries at every compact center. We represent the varying norms in a fixed Banach space and use a partition of unity to combine the local maps. Orthogonal changes of coordinates are retained as compact group actions. Hyperspaces modulo these actions provide absolute retracts through which maps into $\mathcal{M}$ can be approximated with small homotopy tracks. Hanner's domination theorem, stated in Theorem 10.3, then establishes the ANR property. His category comparison in Theorem 10.4 extends this conclusion to all metrizable domains. Since $\mathcal{M}$ is contractible, Theorem 10.5 implies that $\mathcal{M}$ is an AE for metrizable spaces.

Dependence on the other parts. We use the local approximation theorem Theorem 8.8 of Part II. Beyond the common preliminaries, no other result of Part II and no result directly from Part I is used. The dependence on the invariant measures is through the proof of Theorem 8.8. No result of Part IV is needed here. Part IV uses the absolute retract property established in this part to identify $\mathcal{M}$ with Hilbert space.

Organization. In Section 10, we recall the equivariant hyperspace results and the domination and extension theorems used in the proof. In Subsection 11.1, we represent varying finite-dimensional norms isometrically in a fixed Banach space, with compatibility under orthogonal changes of coordinates. In Subsection 11.2, we construct absolute retracts from hyperspaces modulo compact group actions and compare their orbit metrics with the Gromov–Hausdorff distance. In Subsection 11.3, we use a partition of unity to combine local models into global approximations with small homotopy tracks. In Subsection 11.4, we apply Hanner's Theorems 10.3 and 10.4 and use contractibility to obtain extension from closed subsets of arbitrary metrizable spaces, proving Theorem 11.4. **Conventions and notation.**

All compact metric spaces are nonempty, and Banach and Hilbert spaces are real. We identify compact metric spaces with their isometry classes in $\mathcal{M}$. An isometry is a surjective isometric embedding. For a metric space (X, d), we write $\mathcal{K}(X)$ for its nonempty compact subsets, equipped with the Hausdorff metric $\mathsf{H}[d]$. Isometrically embedded compact spaces are identified with their images and carry the restricted ambient metrics. The notation id denotes the identity map on its specified domain. Sequences are indexed by $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

10. Preliminaries

We recall the equivariant hyperspace results and the domination and extension theorems used in this part. The terms AR, ANR, AE, and ANE have the meanings fixed in Definition 2.21.

10.1. **Equivariant hyperspaces and orbit spaces.** A metrizable G-space is a metrizable space with a continuous action of the topological group G. It is a G-AR if every closed equivariant embedding into a metrizable G-space admits an equivariant retraction. A space E is *locally continuum-connected* if for every $x \in E$ and every open set $O \subset E$ with $x \in O$, there is an open set V such that

$$x \in V \subset O, \qquad \forall y \in V \quad \exists C \subset E \quad \begin{cases} C \text{ is compact and connected,} \\ \{x, y\} \subset C \subset O. \end{cases}$$

In a Banach space we can take V to be a sufficiently small open ball and use the segments $C = \{(1-t)x + ty \mid t \in [0,1]\}$.

The following hyperspace theorem is [3, Proposition 3.1], with the corrigendum [4]. It is also stated in [5, Theorem 2.3].

Theorem 10.1 ([3, Proposition 3.1])**.** *Let a compact group G act continuously on a nonempty connected, locally continuum-connected metrizable space E. Then $\mathcal{K}(E)$, with its Vietoris topology and the induced action $gS = \{gv \mid v \in S\}$, is a G-AR.*

Theorem 10.2 ([2, Theorem 8 and Corollary 1])**.** *Let G be a compact group with a countable basis and let E be a metrizable G-AR. Then its orbit space E/G is an AR for metrizable spaces.*

10.2. **Domination and absolute extension.** The next three theorems specify the category changes in the extension argument.

Theorem 10.3 ([17, Theorem 7.2(b)])**.** *Let (T, r) be a separable metrizable space with a fixed compatible metric. Assume that for every $\varepsilon > 0$ there are a separable metrizable ANR $\mathcal{A}$ and continuous maps*

$$\Phi \colon T \to \mathcal{A}, \quad \Psi \colon \mathcal{A} \to T, \quad Q \colon T \times [0,1] \to T$$

such that for every $z \in T$,

$$Q(z, 0) = z, \quad Q(z, 1) = \Psi\Phi(z), \quad \operatorname{diam}_r\{Q(z,t) \mid t \in [0,1]\} < \varepsilon.$$

Then T is an ANR in the category of separable metrizable spaces.

Theorem 10.4 ([18, Theorem 13.4])**.** *A separable metrizable space which is an ANR for separable metrizable spaces is an ANR for all metrizable spaces.*

Theorem 10.5 ([18, Theorem 12.3])**.** *A nonempty contractible metrizable ANE for metrizable spaces is an AE for metrizable spaces.*

11. Absolute extension by global approximation

We now combine the local maps of Theorem 8.8 into a global approximation through an absolute retract. A partition of unity permits arbitrarily many local models to be active, with only local finiteness required. This construction will establish the absolute extension property without a dimension restriction on the extension domain.

11.1. **Isometric representations of varying norms.** The norms N_Y in the local models vary with the compact metric space (Y, e). We first represent all norms of a fixed finite dimension in one Banach space in a way compatible with orthogonal changes of coordinates.

For $n \geq 1$, write

$$\mathbb{S}^{n-1} = \{u \in \mathbb{R}^n \mid \|u\|_2 = 1\}, \qquad E_n = C(\mathbb{S}^{n-1}, \mathbb{R}).$$

We equip E_n with the uniform norm. For a norm N on $\mathbb{R}^n$, its dual norm is

$$N^*(u) = \sup\{|\langle v, u\rangle| \mid N(v) \leq 1\}.$$

The pairing is the Euclidean inner product. In particular, $N^*(u) > 0$ for every $u \neq 0$.

Lemma 11.1. *Let N be a norm on $\mathbb{R}^n$. Then the map $J_N\colon (\mathbb{R}^n, N) \to (E_n, \|\cdot\|_\infty)$ defined by*

$$J_N(v)(u) = \frac{\langle v, u\rangle}{N^*(u)} \qquad (u \in \mathbb{S}^{n-1}) \tag{11.1}$$

is a linear isometric embedding. Thus

$$\|J_N(v) - J_N(w)\|_\infty = N(v - w). \tag{11.2}$$

If norms N_j converge uniformly to N on $\mathbb{S}^{n-1}$ and $v_j \to v$ in $\mathbb{R}^n$, then

$$\|J_{N_j}(v_j) - J_N(v)\|_\infty \to 0.$$

The group $O(n)$ acts continuously on E_n by linear isometries through $(gf)(u) = f(g^{-1}u)$, and

$$J_{N\circ g^{-1}}(gv) = gJ_N(v) \qquad (g \in O(n),\ v \in \mathbb{R}^n). \tag{11.3}$$

Proof. The formula for $J_N(v)$ defines a continuous function because its denominator is positive on the compact sphere. It is linear in v. The definition of the dual norm implies $|\langle v, u\rangle| \leq N(v)N^*(u)$. Hence $\|J_N(v)\|_\infty \leq N(v)$. For $v \neq 0$, the Hahn–Banach theorem [30, Theorem 11.19 and Proposition 11.20] provides a linear functional $\ell\colon \mathbb{R}^n \to \mathbb{R}$ with dual norm one and $\ell(v) = N(v)$. Write $\ell(w) = \langle w, u_0\rangle$ and put $u = u_0/\|u_0\|_2$. Then

$$\frac{|\langle v, u\rangle|}{N^*(u)} = \frac{|\ell(v)|}{N^*(u_0)} = N(v).$$

Equality is immediate for $v = 0$. Linearity now proves (11.2).

To verify continuity, put

$$m = \min_{\mathbb{S}^{n-1}} N > 0, \qquad \varepsilon_j = \sup_{\mathbb{S}^{n-1}} |N_j - N|, \qquad t_j = \varepsilon_j/m.$$

By homogeneity, for large j we have

$$(1 - t_j)N(v) \leq N_j(v) \leq (1 + t_j)N(v) \qquad (v \in \mathbb{R}^n).$$

The inclusions of unit balls therefore imply

$$(1 + t_j)^{-1}N^*(u) \leq N_j^*(u) \leq (1 - t_j)^{-1}N^*(u).$$

Thus the dual norms N_j^* converge uniformly to N^* on the sphere $\mathbb{S}^{n-1}$ and have a common positive lower bound there. Consequently,

$$\begin{aligned}\|J_{N_j}(v_j) - J_N(v)\|_\infty \le \frac{\|v_j - v\|_2}{\min_{\mathbb{S}^{n-1}} N_j^*} \\ + \|v\|_2 \sup_{\mathbb{S}^{n-1}} |(N_j^*)^{-1} - (N^*)^{-1}| \longrightarrow 0.\end{aligned}$$

For later use, the same estimate with a fixed vector implies that for every $M > 0$,

$$\sup_{\|v\|_2 \le M} \|J_{N_j}(v) - J_N(v)\|_\infty \le M \sup_{\mathbb{S}^{n-1}} |(N_j^*)^{-1} - (N^*)^{-1}| \longrightarrow 0. \tag{11.4}$$

Orthogonal transformations preserve the Euclidean sphere, so $\|gf\|_\infty = \|f\|_\infty$. If $g_j \to g$ and $f_j \to f$ uniformly, then

$$\|g_j f_j - gf\|_\infty \le \|f_j - f\|_\infty + \sup_{u \in \mathbb{S}^{n-1}} |f(g_j^{-1}u) - f(g^{-1}u)| \to 0$$

by uniform continuity of f. Finally, substituting $v = gw$ in the definition of the dual norm N^* implies

$$(N \circ g^{-1})^*(u) = N^*(g^{-1}u).$$

It follows that

$$J_{N\circ g^{-1}}(gv)(u) = \frac{\langle gv, u\rangle}{N^*(g^{-1}u)} = J_N(v)(g^{-1}u),$$

which proves (11.3). □

11.2. **Absolute retracts from hyperspace orbits.** The preceding representation converts a change of coordinates into an isometry of a fixed Banach space. The global approximation will use compact subsets of such a space modulo this action. We apply Theorems 10.1 and 10.2 to prove that the resulting hyperspace orbit is an absolute retract.

Lemma 11.2. *Let $(E, \|\cdot\|_E)$ be a separable Banach space, and let a compact metrizable group G act continuously on E by linear isometries. Write $\mathsf{d}(v, w) = \|v - w\|_E$ for its induced metric. Then $\mathcal{K}(E)/G$ is a separable metrizable AR. Its quotient topology is induced by the metric*

$$\delta([S], [T]) = \min_{g \in G} \mathsf{H}[\mathsf{d}](S, gT). \tag{11.5}$$

The map

$$\Psi : \mathcal{K}(E)/G \to \mathcal{M}, \qquad \Psi([S]) = (S, \mathsf{d}|_{S\times S}),$$

is well-defined and 1-Lipschitz.

Proof. First the induced action on $\mathcal{K}(E)$ is continuous. To see this, assume that $g_j \to g$ and $\mathsf{H}[\mathsf{d}](S_j, S) \to 0$. Then

$$\mathsf{H}[\mathsf{d}](g_j S_j, gS) \le \mathsf{H}[\mathsf{d}](S_j, S) + \sup_{s \in S} \|g_j s - gs\|_E \to 0.$$

The last supremum tends to zero because the action is continuous and S is compact. Each group element acts isometrically on the hyperspace $\mathcal{K}(E)$. By Theorem 2.1, the Hausdorff distance induces the Vietoris topology on $\mathcal{K}(E)$. Theorem 10.1 therefore applies because E is connected and locally continuum-connected. Thus $\mathcal{K}(E)$ is a G-AR. The compact metrizable group G has a countable basis, so Theorem 10.2 implies that $\mathcal{K}(E)/G$ is an AR in the metrizable category.

We check the metric and topology explicitly. Write δ for the right-hand side of (11.5). The minimum exists because $g \mapsto \mathsf{H}[\mathsf{d}](S, gT)$ is continuous on the compact group G. If $\delta([S],[T]) = 0$, then $S = gT$ for a minimizing g, so the orbits are equal. Symmetry follows by replacing g with g^{-1}. For $g, h \in G$,

$$\mathsf{H}[\mathsf{d}](S, ghU) \le \mathsf{H}[\mathsf{d}](S, gT) + \mathsf{H}[\mathsf{d}](gT, ghU) = \mathsf{H}[\mathsf{d}](S, gT) + \mathsf{H}[\mathsf{d}](T, hU).$$

Taking minimizing g and h proves the triangle inequality.

Let $\mathsf{q}\colon \mathcal{K}(E) \to \mathcal{K}(E)/G$ be the orbit map. For $r > 0$,

$$\mathsf{q}^{-1}\big(B([S], r; \delta)\big) = \bigcup_{g \in G} B(gS, r; \mathsf{H}[\mathsf{d}]).$$

Thus orbit metric balls are open in the quotient topology. Conversely, if O is quotient-open and $[S] \in O$, choose $r > 0$ with $B(S, r; \mathsf{H}[\mathsf{d}]) \subset \mathsf{q}^{-1}(O)$. Whenever $\delta([S],[T]) < r$, some g satisfies $\mathsf{H}[\mathsf{d}](S, gT) < r$. Then $gT \in \mathsf{q}^{-1}(O)$ and $[T] = [gT] \in O$. Hence $B([S], r; \delta) \subset O$. The two topologies agree.

Since E is separable, Theorem 2.1 implies that $\mathcal{K}(E)$ is separable. Its continuous image under q is therefore separable as well.

For each $g \in G$, its restriction $T \to gT$ is an isometry. Consequently Ψ is independent of the representative of an orbit. Moreover,

$$\begin{aligned}\mathcal{GH}(\Psi([S]), \Psi([T])) &= \mathcal{GH}((S, \mathsf{d}|_{S\times S}), (gT, \mathsf{d}|_{gT\times gT})) \\ &\le \mathsf{H}[\mathsf{d}](S, gT).\end{aligned}$$

Taking the minimum over g proves that Ψ is 1-Lipschitz. $\square$

11.3. **Global approximation with small homotopy tracks.** The next theorem assembles the local models. Its auxiliary AR $\mathcal{A}$ will be the orbit space $\mathcal{K}(E)/G$, and it may depend on the desired approximation scale.

Theorem 11.3. *Let $\rho : \mathcal{M} \to (0, \infty)$ be continuous. Then there are a separable metrizable AR $\mathcal{A}$, continuous maps $\Phi : \mathcal{M} \to \mathcal{A}$ and $\Psi : \mathcal{A} \to \mathcal{M}$, a continuous function $\mathcal{E} : \mathcal{M} \to [0, \infty)$ with $\mathcal{E}(Y, e) < \rho(Y, e)$ for every $(Y, e) \in \mathcal{M}$, and a continuous homotopy $Q : \mathcal{M} \times [0, 1] \to \mathcal{M}$ such that*

$$Q((Y, e), 0) = (Y, e), \qquad Q((Y, e), 1) = \Psi\Phi(Y, e), \tag{11.6}$$

and

$$\mathcal{GH}(Q((Y, e), s), Q((Y, e), t)) \le \frac{|s - t|}{2}\mathcal{E}(Y, e) \qquad (s, t \in [0, 1]). \tag{11.7}$$

Proof. We construct the local cover and its Banach space, define the global map, and then prove continuity and the homotopy estimates.

Step 1. Local models and a partition of unity. At every center, apply Theorem 8.8 with an error smaller than the value of ρ there. Continuity of the error and of ρ permits us to shrink the model domain so that its error is less than $\rho(Y,e)$ at every point in that domain. These domains form an open cover of $\mathcal{M}$.

Since $\mathcal{M}$ is separable and metrizable, there is a countable locally finite partition of unity $\{\lambda_i\}_{i\in\mathbb{Z}_{\geq 0}}$ with $\operatorname{supp}\lambda_i \subset U_i$, where each U_i is a domain of one of these local models. Repeated model domains are allowed when taking a refinement. The supports form a locally finite family. Write the data on U_i as $n_i, N_{i,Y}, \mathbf{v}_{i,Y}, q_{i,Y}$, with error $\mathcal{E}_i(Y,e) < \rho(Y,e)$. We interpret all weighted coordinates below as zero when $\lambda_i(Y,e) = 0$, including when $(Y,e) \notin U_i$.

Step 2. A Banach space with a compact group action. Form the separable Banach space and compact metrizable group

$$E = \left(\bigoplus_{i\in\mathbb{Z}_{\geq 0}} E_{n_i}\right)_{\ell^1}, \qquad G = \prod_{i\in\mathbb{Z}_{\geq 0}} O(n_i). \tag{11.8}$$

The norm in E is $\|\{v_i\}_{i\in\mathbb{Z}_{\geq 0}}\|_E = \sum_i \|v_i\|_\infty$. Write

$$\mathsf{d}\colon E\times E \to [0,\infty), \qquad \mathsf{d}(v,w) = \|v-w\|_E$$

for the induced metric. The group G acts coordinatewise by linear isometries. We prove that the action map

$$G\times E \to E, \qquad (g,v) \longmapsto gv$$

is continuous. Indeed, if $g_j \to g$ in the product group and $v = \{v_i\} \in E$, then for each $m \geq 1$,

$$\|g_j v - gv\|_E \leq \sum_{i=0}^{m} \|(g_j)_i v_i - g_i v_i\|_\infty + 2\sum_{i>m} \|v_i\|_\infty.$$

First choose m to control the tail, and then use continuity on the finitely many remaining coordinates. For varying $w_j \to v$, the inequality

$$\|g_j w_j - gv\|_E \leq \|w_j - v\|_E + \|g_j v - gv\|_E \longrightarrow 0$$

proves continuity of the action map on the product $G\times E$.

Step 3. The global map and its error. For each representative (Y,e) and each choice of orthonormal bases for its active local models, define the map $F_Y\colon Y \to E$ by

$$F_Y(y) = \left\{\lambda_i(Y,e) J_{N_{i,Y}}(\mathbf{v}_{i,Y}(y))\right\}_{i\in\mathbb{Z}_{\geq 0}}, \qquad S_Y = F_Y(Y). \tag{11.9}$$

Only finitely many coordinates are nonzero at each (Y,e). Thus F_Y is continuous and S_Y is nonempty and compact. Equation (11.2) and (A1) of Theorem 8.8 imply

$$\|F_Y(y)\|_E \leq 2\operatorname{diam}(Y,e). \tag{11.10}$$

Changing these bases changes S_Y by one element of G, by (11.3). Isometries between representatives of (Y, e) have the same effect by (8.11) and (11.3). Consequently,

$$\Phi : \mathcal{M} \to \mathcal{A} = \mathcal{K}(E)/G, \qquad \Phi(Y, e) = [S_Y],$$

is well-defined on isometry classes. By Lemma 11.2, $\mathcal{A}$ is a separable metrizable AR and we obtain a map $\Psi : \mathcal{A} \to \mathcal{M}$.

Figure 5 summarizes the assembly and the resulting maps.

$$\begin{tikzcd}[column sep=small]
 & \lambda_{i_0}(Y,e)J_{N_{i_0},Y}(\mathbf{v}_{i_0,Y}(y)) \arrow[dr] & \\
y \in Y \arrow[ur] \arrow[dr] & \vdots & F_Y(y) \in E \\
 & \lambda_{i_1}(Y,e)J_{N_{i_1},Y}(\mathbf{v}_{i_1,Y}(y)) \arrow[ur] &
\end{tikzcd}$$

$$\mathcal{M} \xrightarrow{\ \Phi\ } \mathcal{A} = \mathcal{K}(E)/G \xrightarrow{\ \Psi\ } \mathcal{M}$$

$$\Psi\Phi \simeq \mathrm{id}_{\mathcal{M}} \quad \text{via } Q.$$

FIGURE 5. For a fixed compact metric space (Y, e), the active blocks use the same point y. Their joint image is $S_Y = F_Y(Y)$. Changing bases acts on this image by G, so $\Phi(Y, e) = [S_Y]$. The displayed composite is joined to the identity by the homotopy in Step 5 with the track estimate (11.7).

The pseudometric induced by F_Y is

$$q_Y(y, z) = \|F_Y(y) - F_Y(z)\|_E = \sum_i \lambda_i(Y, e) q_{i,Y}(y, z). \tag{11.11}$$

Every coordinate in (11.9) is evaluated at the same point $y \in Y$. It follows that

$$\mathcal{E}(Y, e) = \|e - q_Y\|_\infty \le \sum_{\{i \mid \lambda_i(Y,e)>0\}} \lambda_i(Y, e)\mathcal{E}_i(Y, e) < \rho(Y, e). \tag{11.12}$$

The weights sum to one, so this estimate is independent of the number of active models.

Step 4. Continuity along common correspondences. We next prove continuity and retain the correspondence information needed for the homotopy. Let $(Y_j, e_j) \to (Y, e)$ in $\mathcal{M}$, and choose isometric embeddings into one compact metric space (Z, h). Choose correspondences $R_j \subset Y_j \times Y$ whose ambient distances tend uniformly to zero. Local finiteness of the supports provides an open neighborhood U of (Y, e) such that

$$I_0 = \{i \in \mathbb{Z}_{\ge 0} \mid U \cap \operatorname{supp} \lambda_i \ne \emptyset\} \quad \text{is finite.}$$

For all sufficiently large j, we have $(Y_j, e_j) \in U$. Thus

$$\lambda_i(Y, e) = \lambda_i(Y_j, e_j) = 0 \qquad (i \notin I_0).$$

For each $i \in I_0$ with $\lambda_i(Y,e) > 0$, align its orthonormal bases along the sequence using Theorem 8.8(A2). All these alignments apply to the same R_j. The aligned coordinate maps satisfy

$$\sup_{(y_j,y)\in R_j} \|\mathbf{v}_{i,Y_j}(y_j) - \mathbf{v}_{i,Y}(y)\|_2 \longrightarrow 0.$$

Their values therefore lie in one bounded subset of $\mathbb{R}^{n_i}$, since $\mathbf{v}_{i,Y}(Y)$ is compact. The uniform estimate (11.4), the isometry identity (11.2), and continuity of the weights imply

$$\begin{aligned}\sup_{(y_j,y)\in R_j} \Big\| &\lambda_i(Y_j,e_j) J_{N_{i,Y_j}}\big(\mathbf{v}_{i,Y_j}(y_j)\big) \\ &- \lambda_i(Y,e) J_{N_{i,Y}}\big(\mathbf{v}_{i,Y}(y)\big)\Big\|_\infty \longrightarrow 0.\end{aligned}$$

For an index with $\lambda_i(Y,e) = 0$, the norm of its entire block at (Y_j,e_j) is at most

$$2\lambda_i(Y_j,e_j)\operatorname{diam}(Y_j,e_j) \to 0.$$

The diameters are bounded along a convergent sequence. This estimate also handles the boundary of a model domain. After these choices of bases, we obtain

$$\sup_{(y_j,y)\in R_j} \|F_{Y_j}(y_j) - F_Y(y)\|_E \to 0. \tag{11.13}$$

Put

$$b_j = \sup_{(y_j,y)\in R_j} \|F_{Y_j}(y_j) - F_Y(y)\|_E, \qquad \gamma_j = \sup_{(y_j,y)\in R_j} h(y_j,y).$$

Both numbers tend to zero. For every point of S_{Y_j}, a corresponding point of S_Y is at Banach distance at most b_j. Surjectivity of the projection $R_j \to Y$ provides the reverse point-to-set bound. Consequently,

$$\mathsf{H}[\mathsf{d}](S_{Y_j}, S_Y) \le b_j \longrightarrow 0.$$

The orbit distance is no larger than this Hausdorff distance for the aligned representatives. This proves continuity of Φ. For $(y_j,y),(z_j,z) \in R_j$, the norm triangle inequality implies

$$|q_{Y_j}(y_j,z_j) - q_Y(y,z)| \le 2b_j, \qquad |e_j(y_j,z_j) - e(y,z)| \le 2\gamma_j.$$

The uniform-norm estimate in Lemma 2.9 therefore yields

$$|\mathcal{E}(Y_j,e_j) - \mathcal{E}(Y,e)| \le 2b_j + 2\gamma_j \longrightarrow 0.$$

This proves continuity of $\mathcal{E}$ in (11.12).

Step 5. The interpolating homotopy. It remains to construct a continuous map $Q\colon \mathcal{M}\times[0,1] \to \mathcal{M}$ with the endpoint identities in (11.6) and the estimate in (11.7).

Definition and compactness. For $(Y,e) \in \mathcal{M}$ and $t \in [0,1]$, put

$$\begin{aligned} q_{Y,t} &= (1-t)e + tq_Y, \\ Q((Y,e),t) &= [Y, q_{Y,t}]. \end{aligned} \tag{11.14}$$

We first check that this defines a point of $\mathcal{M}$ independently of the representative (Y, e). Both e and q_Y are continuous pseudometrics on the compact space Y. Their nonnegative linear combination $q_{Y,t}$ is therefore a continuous pseudometric. Its metric quotient is nonempty and compact by Subsection 2.2. For $0 \leq t < 1$ and distinct $y, z \in Y$, we also have

$$q_{Y,t}(y, z) \geq (1 - t)e(y, z) > 0.$$

Thus $q_{Y,t}$ is a metric when $t < 1$. Its continuity implies that the identity map

$$\mathrm{id}_Y : (Y, e) \to (Y, q_{Y,t})$$

is a continuous bijection from a compact space to a Hausdorff space, and hence a homeomorphism.

By Step 3, q_Y is independent of the orthonormal bases used in the local models. If $u: (Y, e) \to (X, d)$ is an isometry, then its compatibility with the model pseudometrics implies

$$\begin{aligned} &q_{X,t}(u(y), u(z)) \\ &\qquad = (1 - t)e(y, z) + tq_Y(y, z) \\ &\qquad = q_{Y,t}(y, z). \end{aligned}$$

Consequently, the map

$$[Y, q_{Y,t}] \to [X, q_{X,t}], \qquad [y] \longmapsto [u(y)]$$

is an isometry. Thus Q is well defined on $\mathcal{M} \times [0, 1]$.

The two endpoints. We next prove (11.6). The definition implies

$$q_{Y,0} = e, \qquad q_{Y,1} = q_Y.$$

At the second endpoint, (11.11) implies

$$q_Y(y, z) = 0 \quad \Longleftrightarrow \quad F_Y(y) = F_Y(z).$$

Since $S_Y = F_Y(Y)$, the map

$$[Y, q_Y] \to (S_Y, \mathsf{d}|_{S_Y \times S_Y}), \qquad [y] \longmapsto F_Y(y)$$

is a bijection. It is an isometry because

$$q_Y(y, z) = \|F_Y(y) - F_Y(z)\|_E.$$

Hence the definitions of Φ and Ψ imply the following equalities in $\mathcal{M}$.

$$\begin{aligned} Q((Y, e), 0) &= [Y, e] = (Y, e), \\ Q((Y, e), 1) &= [Y, q_Y] \\ &\qquad = (S_Y, \mathsf{d}|_{S_Y \times S_Y}) = \Psi\Phi(Y, e). \end{aligned}$$

Variation of the time parameter. Fix (Y, e). To prove (11.7), let $s, t \in [0, 1]$. By (11.12),

$$\begin{aligned} q_{Y,s} - q_{Y,t} &= (s - t)(q_Y - e), \\ \|q_{Y,s} - q_{Y,t}\|_\infty &= |s - t|\mathcal{E}(Y, e). \end{aligned}$$

Applying (2.2) on the common carrier Y, we obtain

$$\begin{aligned}&\mathcal{GH}\big(Q((Y,e),s),Q((Y,e),t)\big)\\&\qquad\le \tfrac{1}{2}\|q_{Y,s}-q_{Y,t}\|_\infty=\tfrac{1}{2}|s-t|\mathcal{E}(Y,e).\end{aligned}$$

This is (11.7).

Continuity when the space and time both vary. Let

$$(Y_j,e_j)\longrightarrow (Y,e)\quad\text{in } \mathcal{M},\qquad t_j\longrightarrow t\quad\text{in } [0,1].$$

We prove that

$$\mathcal{GH}\big(Q((Y_j,e_j),t_j),Q((Y,e),t)\big)\longrightarrow 0.$$

Use the correspondences R_j and the numbers $\gamma_j,b_j\to 0$ from Step 4. First keep the time equal to t on both spaces. For $(y_j,y),(z_j,z)\in R_j$, the estimates in Step 4 imply

$$\begin{aligned}&\big|q_{Y_j,t}(y_j,z_j)-q_{Y,t}(y,z)\big|\\&\qquad\le (1-t)|e_j(y_j,z_j)-e(y,z)|\\&\qquad\quad+t|q_{Y_j}(y_j,z_j)-q_Y(y,z)|\\&\qquad\le 2(1-t)\gamma_j+2tb_j.\end{aligned}$$

Lemma 2.9 therefore yields

$$\begin{aligned}&\mathcal{GH}\big(Q((Y_j,e_j),t),Q((Y,e),t)\big)\\&\qquad\le (1-t)\gamma_j+tb_j\longrightarrow 0.\end{aligned}$$

This estimate also holds at $t=1$, where the pseudometrics may identify distinct points.

Now allow the time to vary. The continuity proved in Step 4 implies

$$\mathcal{E}(Y_j,e_j)\longrightarrow \mathcal{E}(Y,e)<\infty.$$

By the triangle inequality and (11.7),

$$\begin{aligned}&\mathcal{GH}\big(Q((Y_j,e_j),t_j),Q((Y,e),t)\big)\\&\qquad\le \mathcal{GH}\big(Q((Y_j,e_j),t_j),Q((Y_j,e_j),t)\big)\\&\qquad\quad+\mathcal{GH}\big(Q((Y_j,e_j),t),Q((Y,e),t)\big)\\&\qquad\le \tfrac{1}{2}|t_j-t|\mathcal{E}(Y_j,e_j)+(1-t)\gamma_j+tb_j\longrightarrow 0.\end{aligned}$$

Since $\mathcal{M}\times[0,1]$ is metrizable, this proves continuity of $Q\colon \mathcal{M}\times[0,1]\to\mathcal{M}$, including at the endpoint $t=1$. □

11.4. **Extension from arbitrary metrizable domains.** We now apply the domination and extension theorems from Subsection 10.2.

Theorem 11.4. *The space $\mathcal{M}$ is an absolute extensor for all metrizable spaces. In particular, it is an AR and an ANR in that category.*

Proof. Apply Theorem 11.3 with $\rho \equiv \varepsilon$ for each $\varepsilon > 0$. We obtain continuous maps $\Phi\colon \mathcal{M} \to \mathcal{A}$ and $\Psi\colon \mathcal{A} \to \mathcal{M}$ and a homotopy Q from $\mathrm{id}_{\mathcal{M}}$ to $\Psi \circ \Phi$. By (11.7), for every $(Y, e) \in \mathcal{M}$, we have

$$\mathrm{diam}_{\mathcal{GH}}\{Q((Y,e),t) \mid t \in [0,1]\} \leq \tfrac{1}{2}\mathcal{E}(Y,e) < \tfrac{1}{2}\varepsilon.$$

Theorem 10.3 shows that $\mathcal{M}$ is an ANR in the separable metrizable category.

By Theorem 10.4, $\mathcal{M}$ is an ANR for all metrizable spaces. By the metric ANR/ANE equivalence in Theorem 2.22, it is an ANE in that category.

Lemma 2.20 proves that $\mathcal{M}$ is contractible by scaling every distance to zero. Theorem 10.5 therefore implies that $\mathcal{M}$ is an AE, and hence an AR, for all metrizable spaces. ☐

Proof of Theorem 9.1. The assertion follows from Theorem 11.4. ☐

Part IV. The topology of Gromov–Hausdorff space IV: A Hilbert space characterization

Abstract. We prove that the Gromov–Hausdorff space of isometry classes of nonempty compact metric spaces is homeomorphic to the real separable infinite-dimensional Hilbert space. We construct a discrete approximation for maps from countably many compact metrizable domains by combining metric products with powers of distances. An interval factor and distinct exponents distinguish the images by the Hausdorff dimensions of their connected subsets, while increasing finite equilateral factors prevent accumulation. Together with the absolute retract property established in Part III and the completeness and separability of Gromov–Hausdorff space, this approximation satisfies Toruńczyk's characterization of Hilbert space. As a consequence, every point has a basis of open neighborhoods that strongly deformation retract onto that point.

Keywords. Gromov–Hausdorff space, Hilbert space, discrete approximation property, Hausdorff dimension.

2020 Mathematics Subject Classification. Primary 57N20; Secondary 54E35, 54F45.

12. Introduction to Part IV

The Gromov–Hausdorff space contains compact metric spaces of arbitrary dimension and with different underlying topologies. We determine the topology of the space formed by all these isometry classes. Let $\mathcal{M}$ denote this space with the ordinary unpointed Gromov–Hausdorff distance $\mathcal{GH}$. Antonyan records the questions whether $\mathcal{M}$ is an absolute retract and whether it is homeomorphic to the separable infinite-dimensional Hilbert space [5, p. 2]. For each positive integer n, Antonyan identifies the subspace represented by compact subsets of $\mathbb{R}^n$ with the Hilbert cube minus a point [6, Corollary 5.3].

The author constructed families of branching Gromov–Hausdorff geodesics continuously parametrized by the Hilbert cube [20, Theorem 1.3]. The author also constructed topological embeddings of arbitrary compact metrizable spaces, with finitely many distinct prescribed values, into the subsets of $\mathcal{M}$ consisting of continua [21, Theorem 1.1], spaces with prescribed dimensions [22, Theorem 1.3], and compact metric trees [23, Theorem 1.1], respectively.

The subspaces of $\mathcal{M}$ consisting of finite metric spaces also have a precise dimension theory. Nakajima, Yamauchi, and Zava [32, arXiv v1, Theorem A] prove that the subspace of spaces with at most n points has topological dimension $n(n-1)/2$. Nakajima and Shioya [31, arXiv v1, Main Theorem 1.2] construct a compact metrizable space containing a dense topological copy of $\mathcal{M}$.

The absolute retract property from Part III supplies one hypothesis of Toruńczyk's characterization of Hilbert space. We establish the remaining discrete approximation property and obtain the following theorem.

Theorem 12.1. *The space* $\mathcal{M}$ *is homeomorphic to the real Hilbert space* ℓ^2*.*

For a sequence of compact metrizable domains $\{K_i\}_{i\in\mathbb{Z}_{\geq 0}}$, continuous maps $f_i\colon K_i \to \mathcal{M}$, and an open cover $\mathcal{U}$ of $\mathcal{M}$, we construct continuous maps $g_i\colon K_i \to \mathcal{M}$ that are $\mathcal{U}$-close to f_i and satisfy

$$\{g_i(K_i)\}_{i\in\mathbb{Z}_{\geq 0}} \text{ is discrete in } \mathcal{M}.$$

Thus every point of $\mathcal{M}$ has a neighborhood meeting at most one image. The case where all domains are Hilbert cubes is the approximation condition in Toruńczyk's characterization, stated in Theorem 13.6 and taken from [34, p. 248, assertion (i)], with the correction in [35, Section C]. Together with completeness, separability, and the absolute retract property, this condition proves Theorem 12.1.

The discrete approximation argument uses metric products and powers of distances. Small interval factors and finite equilateral factors keep the approximations close to the original compact metric spaces. Distinct powers distinguish the images by the Hausdorff dimensions of connected subsets, while increasing cardinalities of the finite factors prevent accumulation of the family.

The main theorem also clarifies the local contraction question. We use local contractibility in the sense that every neighborhood U of a point contains a neighborhood V whose inclusion into U is null-homotopic. The homeomorphism in Theorem 12.1 implies the stronger existence of a basis of open neighborhoods which strongly deform onto their centers. This conclusion is topological and does not prescribe the neighborhoods as balls for $\mathcal{GH}$.

Dependence on the other parts. The Hilbert space characterization uses Theorem 9.1 of Part III. Its proof uses no result from Parts I or II directly. Beyond the common preliminaries, the discrete approximation argument uses none of the other parts. Its construction depends only on products, powers of distances, and the topology induced by the Gromov–Hausdorff distance.

Organization. In Section 13, we define discrete families and state the Hilbert space characterization. In Subsection 13.2, we establish estimates for metric products and powers of distances and introduce the connected-subset dimension invariant used to distinguish the approximating spaces. In Subsection 14.1, we construct approximations of maps from countably many compact domains whose images form a discrete family, proving Theorem 14.1. In Subsection 14.2, we apply Toruńczyk's characterization to prove Theorem 12.1 and derive strong local contractions. In Section 15, we conclude with questions about related spaces and the geometry of Gromov–Hausdorff balls.

Conventions and notation. All compact metric spaces are nonempty, and Banach and Hilbert spaces are real. We identify compact metric spaces with their isometry classes in $\mathcal{M}$. An isometry is a surjective isometric embedding. For a metric space (X, d), we write $\mathcal{K}(X)$ for its nonempty compact subsets, equipped with the Hausdorff metric $\mathsf{H}[d]$. Isometrically embedded compact spaces are identified with their images and carry the restricted ambient metrics. The notation id denotes the identity map on its specified domain. Sequences are indexed by $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

13. Preliminaries

We define discrete families and approximation with respect to an open cover. We then establish the product and snowflake estimates used in the construction and recall Toruńczyk's characterization of Hilbert space.

13.1. **Discrete families and close maps.**

To state the approximation condition in Theorem 13.6, we recall discreteness of a family of subsets and closeness of maps with respect to an open cover.

Definition 13.1. A family $\{A_i\}_{i \in I}$ of subsets of a space T is *discrete in* T if every point of T has a neighborhood meeting at most one member of the family $\{A_i\}_{i \in I}$. It is *locally finite* if every point has a neighborhood meeting only finitely many members. These requirements apply also at points outside the union. For an open cover $\mathcal{U}$ of T, two maps $f, g : D \to T$ are $\mathcal{U}$*-close* if for each $z \in D$ some member of $\mathcal{U}$ contains both $f(z)$ and $g(z)$.

13.2. **Products and a distinguishing invariant.**

For compact metric spaces (A, d_0) and (B, d_1), write $d_0 \vee d_1$ for the max metric on $A \times B$, defined by

$$(d_0 \vee d_1)((a, b), (u, v)) = \max\{d_0(a, u), d_1(b, v)\}.$$

For $0 < \beta \leq 1$, the snowflake of (A, d_0) is the metric space (A, d_0^β). This is a metric because $(s + t)^\beta \leq s^\beta + t^\beta$, and it induces the original topology.

Lemma 13.2. *For nonempty compact metric spaces* (A, d_0)*,* (B, d_1)*,* (C, e_0)*,* (D, e_1)*, and* $0 < \beta \leq 1$*, the following inequalities hold.*

$$\begin{aligned} &\mathcal{GH}((A \times B, d_0 \vee d_1), (C \times D, e_0 \vee e_1)) \\ &\qquad \leq \max\{\mathcal{GH}((A, d_0), (C, e_0)), \mathcal{GH}((B, d_1), (D, e_1))\}. \end{aligned} \tag{13.1}$$

$$\mathcal{GH}((A, d_0^\beta), (B, d_1^\beta)) \leq 2^{\beta - 1} \mathcal{GH}((A, d_0), (B, d_1))^\beta. \tag{13.2}$$

Proof. Let $R_A \subset A \times C$ and $R_B \subset B \times D$ be correspondences. By rearranging coordinates, we regard their product $R_\times = R_A \times R_B$ as a correspondence between $A \times B$ and $C \times D$. The difference between two maxima is bounded by the maximum of the differences of their entries. Hence

$$\operatorname{dis} R_\times \leq \max\{\operatorname{dis} R_A, \operatorname{dis} R_B\}.$$

Taking the infimum over the two factor correspondences proves (13.1) by (2.1). For the second assertion, the inequality $|s^\beta - t^\beta| \leq |s - t|^\beta$ shows that a correspondence of distortion r before snowflaking has distortion at most r^β afterwards. Apply (2.1) and take the infimum. $\square$

To distinguish the approximating spaces, we use the infimum of the Hausdorff dimensions of their nondegenerate subcontinua.

Definition 13.3. For a compact metric space (Z,d), define

$$\theta(Z,d)=\inf_C \dim_{\mathrm{H}}(C,d|_{C\times C}), \tag{13.3}$$

where C ranges over compact connected subsets of Z containing at least two points. The infimum of the empty set is ∞ and $\dim_{\mathrm{H}}$ denotes Hausdorff dimension.

Isometries preserve this quantity because they preserve compactness, connectedness, and Hausdorff dimension.

The covering argument for Hölder maps in [16, Propositions 2.2–2.3, pp. 27–29] applies to snowflake metrics as follows.

Theorem 13.4. *For a metric space (X,d) and $0<\beta\le 1$, we have*

$$\dim_{\mathrm{H}}(X,d^\beta)=\frac{\dim_{\mathrm{H}}(X,d)}{\beta}.$$

The formula includes infinite Hausdorff dimension.

Proof. For every subset $U\subset X$, we have

$$\operatorname{diam}_{d^\beta} U=(\operatorname{diam}_d U)^\beta.$$

Let $\{U_k\}_{k\in\mathbb{Z}_{\ge 0}}$ be a countable cover of X. For every $s>0$ and $r>0$, we therefore have

$$\sum_{k\in\mathbb{Z}_{\ge 0}}(\operatorname{diam}_{d^\beta} U_k)^s=\sum_{k\in\mathbb{Z}_{\ge 0}}(\operatorname{diam}_d U_k)^{\beta s}$$

and

$$\sup_k \operatorname{diam}_{d^\beta} U_k\le r \quad\iff\quad \sup_k \operatorname{diam}_d U_k\le r^{1/\beta}.$$

Taking infima over coverings and then letting $r\downarrow 0$ shows that the critical exponent is divided by β. This also applies when the critical exponent is infinite. $\square$

Lemma 13.5. *If a compact metric space (Z,d) contains an isometric nondegenerate interval, then $\theta(Z,d)=1$. For every compact metric space (Z,d) and every $0<\beta\le 1$, we have*

$$\theta(Z,d^\beta)=\frac{\theta(Z,d)}{\beta}.$$

Proof. Every connected metric space containing two distinct points has Hausdorff dimension at least one. Indeed, the distance from one of those points is a 1-Lipschitz function whose image contains a nondegenerate interval. Hausdorff dimension does not increase under Lipschitz maps, as follows by applying such a map to covers in its definition. An isometric interval attains this lower bound, which proves the first assertion.

Snowflaking preserves the topology, and hence the collection of nondegenerate compact connected subsets. Theorem 13.4 implies

$$\dim_{\mathrm{H}}(C,(d|_{C\times C})^{\beta}) = \frac{\dim_{\mathrm{H}}(C, d|_{C\times C})}{\beta},$$

including infinite dimension. Taking the infimum over the same compact connected subsets proves the second assertion, also when that collection is empty. □

13.3. **A Hilbert space characterization.** We use the following characterization of Hilbert space. It is Toruńczyk's theorem [34, p. 248, condition (i)], with its correction [35, Section C]. The correction repairs the proof and leaves this characterization unchanged.

Theorem 13.6 (Toruńczyk, [34, p. 248, condition (i)], [35, Section C])**.** *Let (T, r) be a nonempty complete separable metric AR, and let $Q = [0,1]^{\mathbb{Z}_{\geq 0}}$ be the Hilbert cube. Then T is homeomorphic to real ℓ^2 if and only if, for every continuous map $f : \mathbb{Z}_{\geq 0} \times Q \to T$ and every open cover $\mathcal{U}$ of T, there is a continuous $\mathcal{U}$-close map $g : \mathbb{Z}_{\geq 0} \times Q \to T$ such that $\{g(\{i\} \times Q)\}_{i\in\mathbb{Z}_{\geq 0}}$ is discrete in T. The nonnegative integers have the discrete topology.*

14. Discrete approximation and Hilbert space topology

The remaining condition in Toruńczyk's characterization of Hilbert space is an approximation property for maps from countably many Hilbert cubes. We prove a version allowing arbitrary compact metrizable domains. Small metric products provide increasingly large separated finite subsets, while distinct snowflake exponents distinguish the resulting image sets.

14.1. **Discrete approximation for compact domains.** We now construct the approximating maps and prove discreteness at every point of the ambient Gromov–Hausdorff space.

Theorem 14.1. *Let $\{K_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be compact metrizable spaces, let $f_i : K_i \to \mathcal{M}$ be continuous, and let $\mathcal{U}$ be an open cover of $\mathcal{M}$. Then there are continuous maps $g_i : K_i \to \mathcal{M}$ such that each g_i is $\mathcal{U}$-close to f_i and the family $\{g_i(K_i)\}_{i\in\mathbb{Z}_{\geq 0}}$ is discrete in $\mathcal{M}$.*

Proof. Empty domains may be omitted and their empty maps restored at the end. If all domains are empty, there is nothing to prove. Index the nonempty domains by an initial segment I of the nonnegative integers, which may be finite.

Since $\mathcal{M}$ is metrizable, by Theorem 2.10 and rescaling we obtain a continuous pseudometric $\mathcal{R}\colon \mathcal{M} \times \mathcal{M} \to [0,\infty)$ such that the family

$$\{\mathfrak{B}((X,d),1;\mathcal{R})\}_{(X,d)\in\mathcal{M}}$$

refines $\mathcal{U}$, where

$$\mathfrak{B}((X,d),r;\mathcal{R}) = \{(Y,e)\in\mathcal{M} \mid \mathcal{R}((X,d),(Y,e)) < r\} \quad (r>0).$$

Equip $\mathcal{M}\times\mathcal{M}$ with the maximum product metric $\mathcal{GH}\vee\mathcal{GH}$, and put

$$\mathcal{A} = \{((X,d),(Y,e))\in\mathcal{M}\times\mathcal{M} \mid \mathcal{R}((X,d),(Y,e))\geq 1\}.$$

The set $\mathcal{A}$ is closed and disjoint from the diagonal. Define functions $L\colon\mathcal{M}\to(0,1]$ and $a\colon\mathcal{M}\to(0,1/16]$ by

$$\begin{aligned} L(X,d) &= \min\left\{1, \operatorname{dist}_{\mathcal{GH}\vee\mathcal{GH}}\big(((X,d),(X,d)),\mathcal{A}\big)\right\}, \\ a(X,d) &= \frac{L(X,d)}{16}. \end{aligned} \tag{14.1}$$

If $\mathcal{A}=\emptyset$, we set $L\equiv 1$. The function L is positive because $\mathcal{A}$ is closed and disjoint from the diagonal. Since the map $(X,d)\mapsto((X,d),(X,d))$ is an isometric embedding and distance to a nonempty set is 1-Lipschitz, L is 1-Lipschitz. Consequently,

$$0 < a(X,d) \leq \frac{1}{16}, \qquad |a(X,d)-a(Y,e)| \leq \frac{1}{16}\mathcal{GH}((X,d),(Y,e)). \tag{14.2}$$

If $\mathcal{GH}((X,d),(Y,e)) < a(X,d)$, then

$$\begin{aligned} &(\mathcal{GH}\vee\mathcal{GH})\big(((X,d),(X,d)),((X,d),(Y,e))\big) \\ &\quad = \mathcal{GH}((X,d),(Y,e)) < L(X,d). \end{aligned}$$

Thus $\mathcal{R}((X,d),(Y,e)) < 1$, and both spaces belong to a common member of $\mathcal{U}$.

For $m\in\mathbb{Z}_{\geq 1}$, let $E_m=\{0,\dots,m-1\}$, and let $d_{m,r}$ be its equilateral metric with off-diagonal distance $r>0$. Intervals carry the Euclidean metric $d_{\mathbb{R}}(s,t)=|s-t|$. For $i\in I$, set $m_i=i+1$ and define a map $P_i\colon\mathcal{M}\to\mathcal{M}$ by

$$\begin{aligned} P_i(X,d) &= (X\times[0,r]\times E_{m_i}, d\vee d_{\mathbb{R}}\vee d_{m_i,r}), \\ r &= a(X,d). \end{aligned} \tag{14.3}$$

The projection onto X defines a correspondence of distortion at most $a(X,d)$. Hence

$$\mathcal{GH}(P_i(X,d),(X,d)) \leq \frac{a(X,d)}{2}. \tag{14.4}$$

The interval and equilateral factors are scalings of fixed compact spaces for each i. Continuity of scaling and (13.1) show that P_i is continuous.

Since the set $F_i=f_i(K_i)$ is compact and the function a is positive and continuous, a attains a positive minimum on F_i. Moreover, the function $(X,d)\mapsto\operatorname{diam}(X,d)$ is bounded on F_i. Put

$$r_i = \min_{(X,d)\in F_i} a(X,d) > 0, \qquad M_i = \max\left\{1, \max_{(X,d)\in F_i}\operatorname{diam}(X,d)\right\}.$$

The diameter of every $P_i(X,d)$ with $(X,d)\in F_i$ is at most M_i. Choose strictly increasing positive integers k_i such that, with $\beta_i = 1-2^{-k_i}$,

$$\sup_{0\leq t\leq M_i} |t^{\beta_i}-t| < r_i. \tag{14.5}$$

Such a recursive choice is possible because $(\beta, t) \mapsto t^\beta$ is continuous on $[1/2, 1] \times [0, M_i]$, including at $t = 0$. In particular, the exponents $\{\beta_i\}_{i \in I}$ are pairwise distinct and belong to $[1/2, 1)$.

Define $g_i \colon K_i \to \mathcal{M}$ by

$$g_i(z) = (A, h^{\beta_i}) \quad \text{when } (A, h) = P_i(f_i(z)), \qquad z \in K_i. \tag{14.6}$$

These maps are continuous by Lemma 13.2. The identity correspondence before and after snowflaking, (14.5), and (14.4) imply that, for $(X, d) = f_i(z)$,

$$\mathcal{GH}(g_i(z), (X, d)) < \frac{r_i}{2} + \frac{a(X, d)}{2} \leq a(X, d). \tag{14.7}$$

Thus g_i and f_i are $\mathcal{U}$-close. Each image $g_i(K_i)$ is compact and hence closed in $\mathcal{M}$.

For every $(X, d) \in \mathcal{M}$, the space $P_i(X, d)$ contains an isometric interval $\{x\} \times [0, a(X, d)] \times \{\zeta\}$, where $x \in X$ and $\zeta \in E_{m_i}$ may be arbitrary. Lemma 13.5 therefore implies

$$\theta(P_i(X, d)) = 1, \qquad \theta(g_i(z)) = \frac{1}{\beta_i}. \tag{14.8}$$

The image sets $g_i(K_i)$ are consequently pairwise disjoint.

Figure 6 separates the roles of the factors and exponents.

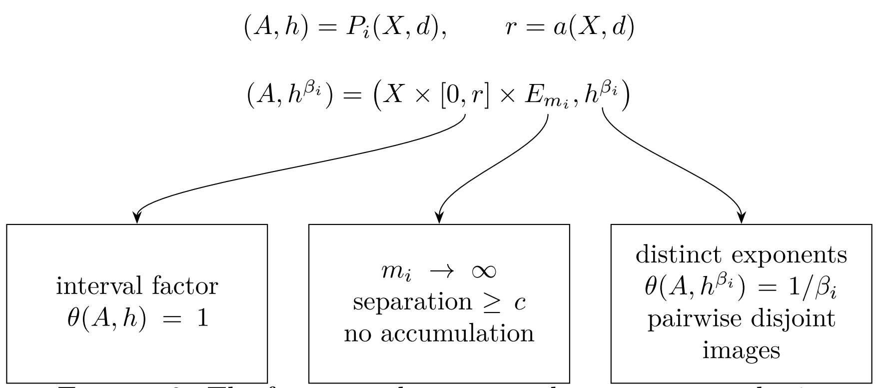

Figure 6. The factors and exponents have separate roles in the discrete approximation. The maximum product metric is $h = d \vee d_{\mathbb{R}} \vee d_{m_i, r}$, where $d_{m_i, r}$ is the equilateral metric with off-diagonal distance r. Under a hypothetical convergent sequence from distinct images, the scales have a positive lower bound $c > 0$. The equilateral factors then contradict the uniform bound on the cardinalities of c-separated subsets.

It remains to prove local finiteness. For the sake of contradiction, suppose that the family $\{g_i(K_i)\}_{i \in I}$ is not locally finite at $(Z, h_\infty) \in \mathcal{M}$. There are strictly increasing indices i_j and points $z_j \in K_{i_j}$ such that

$$(Y_j, e_j) = g_{i_j}(z_j) \to (Z, h_\infty).$$

Set $(X_j, d_j) = f_{i_j}(z_j)$ and $a_j = a(X_j, d_j)$. Equations (14.2) and (14.7) imply

$$a(Z, h_\infty) \le a_j + \frac{1}{16}\mathcal{GH}((X_j, d_j), (Z, h_\infty)) \le \frac{17}{16}a_j + \frac{1}{16}\mathcal{GH}((Y_j, e_j), (Z, h_\infty)). \tag{14.9}$$

Hence, for all sufficiently large j, $a_j \ge c = a(Z, h_\infty)/2 > 0$.

We use the product representative of (Y_j, e_j) from (14.3) and (14.6). Choose $x_j \in X_j$ and put

$$y_{j,\zeta} = (x_j, 0, \zeta) \in Y_j \qquad (\zeta \in E_{m_{i_j}}).$$

The set

$$S_j = \{y_{j,\zeta} \mid \zeta \in E_{m_{i_j}}\} \subset Y_j$$

has cardinality $|S_j| = m_{i_j}$. For all sufficiently large j and all distinct $\zeta_0, \zeta_1 \in E_{m_{i_j}}$, we have

$$\begin{aligned} e_j(y_{j,\zeta_0}, y_{j,\zeta_1}) &= \Big(\max\{d_j(x_j, x_j), d_{\mathbb{R}}(0,0), d_{m_{i_j}, a_j}(\zeta_0, \zeta_1)\}\Big)^{\beta_{i_j}} \\ &= (\max\{0, 0, a_j\})^{\beta_{i_j}} \\ &= a_j^{\beta_{i_j}} \ge a_j \ge c, \end{aligned}$$

since $0 < a_j \le 1$ and $0 < \beta_{i_j} \le 1$. Thus S_j is c-separated for all sufficiently large j. Since $(Y_j, e_j) \to (Z, h_\infty)$, Corollary 2.5 provides a common upper bound for $|S_j|$. This contradicts

$$|S_j| = m_{i_j} = i_j + 1 \longrightarrow \infty.$$

Thus the family $\{g_i(K_i)\}_{i \in I}$ is locally finite. A locally finite family of pairwise disjoint closed sets is discrete, since the union of a locally finite family of closed sets is closed [15, Corollary 1.1.12]. This argument also covers a finite index set I, for which local finiteness is automatic. $\square$

14.2. **Proof of the Hilbert space theorem.** We apply Theorem 13.6 to $\mathcal{M}$ using the AR property from Part III and the discrete approximation in Theorem 14.1.

Proof of Theorem 12.1. Lemma 2.6 proves that $\mathcal{M}$ is complete and separable. It is nonempty and is an AR by Theorem 9.1 of Part III. Let $f \colon \mathbb{Z}_{\ge 0} \times Q \to \mathcal{M}$ be continuous and let $\mathcal{U}$ be an open cover of $\mathcal{M}$. For each $i \in \mathbb{Z}_{\ge 0}$, put

$$K_i = Q, \qquad f_i \colon Q \to \mathcal{M}, \qquad f_i(z) = f(i, z).$$

Theorem 14.1 provides continuous maps $g_i \colon Q \to \mathcal{M}$ that are $\mathcal{U}$-close to f_i and whose images form a discrete family in $\mathcal{M}$. Define

$$g \colon \mathbb{Z}_{\ge 0} \times Q \to \mathcal{M}, \qquad g(i, z) = g_i(z).$$

Each subset $\{i\} \times Q$ is open in the domain, so g is continuous. It is $\mathcal{U}$-close to f, and

$$\{g(\{i\} \times Q)\}_{i \in \mathbb{Z}_{\ge 0}} = \{g_i(Q)\}_{i \in \mathbb{Z}_{\ge 0}} \quad \text{is discrete in } \mathcal{M}.$$

Thus the hypotheses of Theorem 13.6 hold. It follows that $\mathcal{M}$ is homeomorphic to real ℓ^2. $\square$

The main theorem also implies strong local contractions.

Corollary 14.2. *Every point $(X, d) \in \mathcal{M}$ has a basis of open neighborhoods which strongly deformation retract onto $\{(X, d)\}$.*

Proof. Fix a homeomorphism $f \colon \mathcal{M} \to \ell^2$ and a point $(X, d) \in \mathcal{M}$. For each $r > 0$, let

$$O_r = \{(Y, e) \in \mathcal{M} \mid \|f(Y, e) - f(X, d)\|_2 < r\}.$$

These sets form a neighborhood basis because open balls form a basis in ℓ^2 and f is a homeomorphism. On $O_r \times [0, 1]$, define

$$H((Y, e), t) = f^{-1}\big((1 - t)f(Y, e) + tf(X, d)\big).$$

This is continuous. The vector inside f^{-1} is at distance

$$(1 - t)\|f(Y, e) - f(X, d)\|_2 < r$$

from the center, so the homotopy stays in O_r. For every $(Y, e) \in O_r$ and every $t \in [0, 1]$, we have

$$\begin{aligned} H((Y, e), 0) &= (Y, e), \\ H((Y, e), 1) &= (X, d), \\ H((X, d), t) &= (X, d). \end{aligned}$$

These identities prove the strong deformation retraction. $\square$

15. Questions

We conclude with questions about related spaces and the geometry of GH balls. We use the mm-isomorphism classes and box topology of Definition 2.18.

Question 15.1. Is the space of mm-isomorphism classes of metric measure spaces, equipped with the box topology, homeomorphic to real ℓ^2?

We recall the concentration topology to distinguish it from the box topology. For measurable real functions f, g on I, identified up to equality almost everywhere, define the Ky Fan distance by

$$d_{\mathrm{KF}}(f, g) = \inf\{\varepsilon > 0 \mid \mathcal{L}^1(\{s \in I \mid |f(s) - g(s)| > \varepsilon\}) \le \varepsilon\}.$$

Let $\mathrm{Lip}_1(X)$ be the real 1-Lipschitz functions on X. For a parameter φ of X, put $\varphi^* \mathrm{Lip}_1(X) = \{f \circ \varphi \mid f \in \mathrm{Lip}_1(X)\}$. The observable distance is

$$d_{\mathrm{conc}}(X, Y) = \inf_{\varphi, \psi} \mathsf{H}[d_{\mathrm{KF}}]\big(\varphi^* \mathrm{Lip}_1(X), \psi^* \mathrm{Lip}_1(Y)\big),$$

where the infimum runs over parameters of X and Y. Here $\mathsf{H}[d_{\mathrm{KF}}]$ uses the same formula for Hausdorff distance as for compact subsets, applied also to subsets that need not be compact. It is finite because $d_{\mathrm{KF}} \le 1$. The metric

d_{conc} on $\mathcal{X}$ induces the *concentration topology* [26, arXiv v1, Definitions 2.4, 2.13 and Theorem 2.14].

For the concentration topology, the analogous question has a negative answer. Indeed, Kazukawa, Nakajima, and Shioya prove that $(\mathcal{X}, d_{\mathrm{conc}})$ is not a Baire space [26, arXiv v1, Theorem 1.3], whereas ℓ^2 is a Baire space.

Nakajima and Shioya construct a compact metrizable compactification of $\mathcal{M}$ [31, arXiv v1, Main Theorem 1.2]. This leads to the following question.

Question 15.2. Is the compactification of $\mathcal{M}$ constructed by Nakajima and Shioya homeomorphic to the Hilbert cube Q?

The neighborhood basis in Corollary 14.2 need not consist of GH balls. We use the notation $\mathfrak{B}(X, r)$ for open Gromov–Hausdorff balls introduced in Section 2.

Question 15.3. For every $X \in \mathcal{M}$, does there exist $r_0 > 0$ such that, for every $0 < r < r_0$, the ball $\mathfrak{B}(X, r)$ is contractible? More strongly, can each such ball be strongly deformation retracted onto $\{X\}$?

One can also ask for a contraction whose image lies in a larger ball with a controlled radius.

Question 15.4. For every $X \in \mathcal{M}$, do there exist $C \geq 1$ and $r_0 > 0$ such that, for every $0 < r < r_0$, there is a continuous map

$$H\colon \mathfrak{B}(X, r) \times [0, 1] \to \mathfrak{B}(X, Cr)$$

satisfying

$$H(Y, 0) = Y, \qquad H(Y, 1) = X, \qquad H(X, t) = X$$

for all $Y \in \mathfrak{B}(X, r)$ and $t \in [0, 1]$? Can C be chosen independently of X, with r_0 still allowed to depend on X?

To specify the final compactification, use $\mathcal{X}$ and $\square$ from Definition 2.18. Write $Y \prec X$ when a measure-preserving 1-Lipschitz map from the support of X to the support of Y exists. A pyramid is a nonempty box-closed subset $\mathcal{P} \subset \mathcal{X}$ such that

$$X \in \mathcal{P}, \quad Y \prec X \quad \Longrightarrow \quad Y \in \mathcal{P},$$

and any two members are dominated by a third member of $\mathcal{P}$ [26, arXiv v1, Definitions 2.2 and 2.16]. Let Π be the set of pyramids. Its weak topology is metrizable and is characterized by $\mathcal{P}_n \to \mathcal{P}$ if and only if

$$\begin{aligned} \square(X, \mathcal{P}_n) &\longrightarrow 0 \quad && (X \in \mathcal{P}), \\ \liminf_{n\to\infty} \square(X, \mathcal{P}_n) &> 0 && (X \in \mathcal{X} \setminus \mathcal{P}), \end{aligned} \qquad \square(X, \mathcal{P}_n) = \inf_{Y \in \mathcal{P}_n} \square(X, Y).$$

The assignment $X \mapsto \{Y \in \mathcal{X} \mid Y \prec X\}$ identifies the concentration space $(\mathcal{X}, d_{\mathrm{conc}})$ with a dense subspace of the compact space Π [26, arXiv v1, Definition 2.17 and Theorem 2.18]. Kazukawa, Nakajima, and Shioya prove that Π is contractible and locally path connected [26, arXiv v1, Theorems 1.4 and 1.6]. This is a different compactification from the one in Question 15.2.

Question 15.5. Is Π with the weak topology homeomorphic to the Hilbert cube Q?

Diameter normalization suggests another natural subspace of $\mathcal{M}$.

Question 15.6. Is the subspace

$$\{(X, d) \in \mathcal{M} \mid \operatorname{diam}(X, d) = 1\},$$

equipped with the Gromov–Hausdorff topology, homeomorphic to real ℓ^2?

Corollary 5.7 embeds the Gromov–Hausdorff space $\mathcal{M}$ as a retract of the measured space $\mathcal{E}$. We ask whether $\mathcal{E}$ has the same homeomorphism type as $\mathcal{M}$.

Question 15.7. Is the space $\mathcal{E}$ of isomorphism classes of nonempty compact metric spaces equipped with Borel probability measures, without a full-support assumption, equipped with the Gromov–Hausdorff–Prokhorov distance $\mathcal{GHP}$ of Definition 4.1, homeomorphic to real ℓ^2?

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